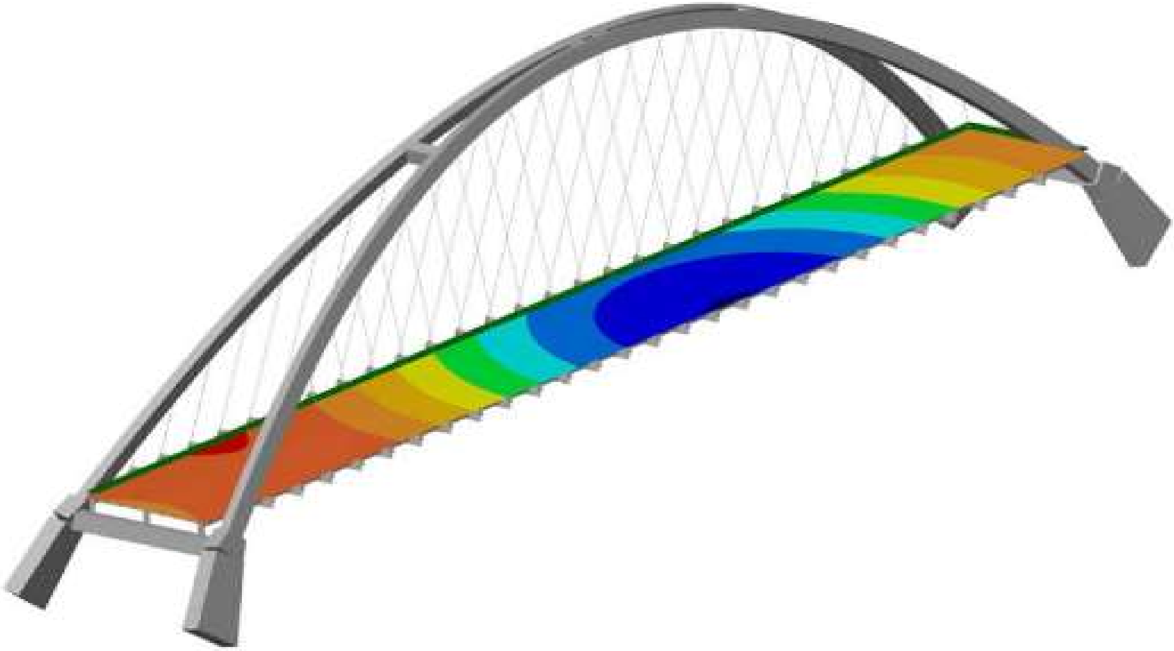
INTRODUCTION TO REFINED ANALSIS FOR HIGHWAY BRIDGES

PARTICIPANT WORKBOOK



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June 28-30, 2022
Missouri DOT

U.S. Department of Transportation
Federal Highway Administration

Federal Highway Administration
RESOURCE CENTER
Office of Innovation Implementation

Refined Analysis for Bridge Structures
Workshop Agenda for MODOT
June 28-30, 2022

Tuesday (6/28) 8:00 am – noon (CDT)

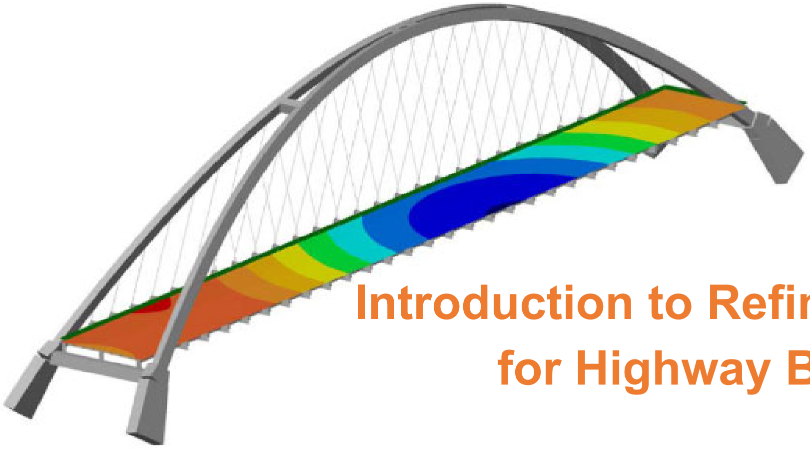
- 08:00 Welcoming remarks by instructors & host agency
- 08:10 Lesson 0 - Introduction
- 09:00 Lesson 1 - Fundamentals of Finite Element Analysis (FEA) & Modeling (Part 1)
- 10:15 Short break
- 10:30 Lesson 1 - Fundamentals of Finite Element Analysis (FEA) & Modeling (Part 2)
- Q & A
- 12:00 Adjourn

Wednesday (6/29) 8:00 am – noon (CDT)

- 08:00 Lesson 2 - General Girder Bridge Modeling
- 09:30 Short break
- 09:40 Lesson 3 - Modeling Special Topics
- 10:40 Lesson 4 - Load Applications w/FEA software demo
- Q & A
- 12:00 Adjourn

Thursday (6/30) 8:00 am – noon (CDT)

- 08:00 Lesson 5 - Problem set-up and analysis procedures using P/S girder bridge example
- 09:30 Short break
- 09:40 Lesson 6 - Verification/Validation of results
- 10:40 Analysis to Design w/FEA software demo
- 11:40 Course closeout
- Q & A/Course Evaluation
- Closing remarks by host agency
- 12:00 Adjourn



Introduction to Refined Analysis for Highway Bridges

Missouri Dept of Transportation (MODOT)

June 28-30, 2022

U.S. Department of Transportation

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Lesson 0 Introduction

Instructors

- Waider Wong, PE, SE, MSSE.
  - Senior Structural Engineer
  - FHWA Resource Center
  - Waider.Wong@dot.gov
- Jeffrey Ger, PE, PhD.
  - Senior Structural Engineer
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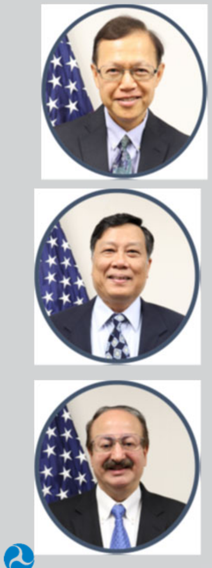
  - Jeffrey.Ger@dot.gov
- Jamal Elkaissi, PE, MS
  - Structural Engineer
  - FHWA Resource Center
  - Jamal.Elkaissi@dot.gov



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Welcome & Self-Introduction

- Please enter your name, position & office, course expectations (short sentence) in the polling.

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Polling Question 1 (LoS5Q1):

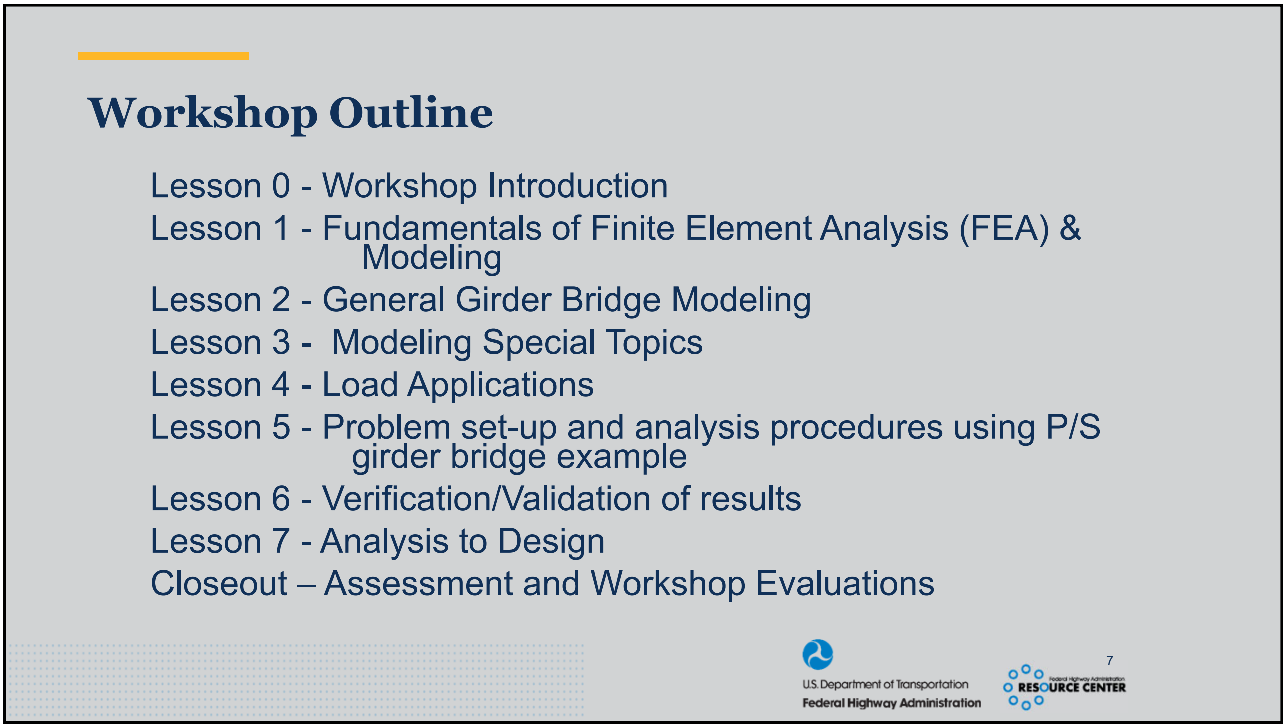
How many years of experience with refined analysis including but not limited to 2D grillage analogy, plates over eccentric beam (PEB) or 3D FEA?

a) 0 or 1 yr
b) 2 - 3 yrs
c) 3 – 5 yrs
d) > 5 yrs
e) > 10 yrs

Polling Question 2 (LoS6Q2)::

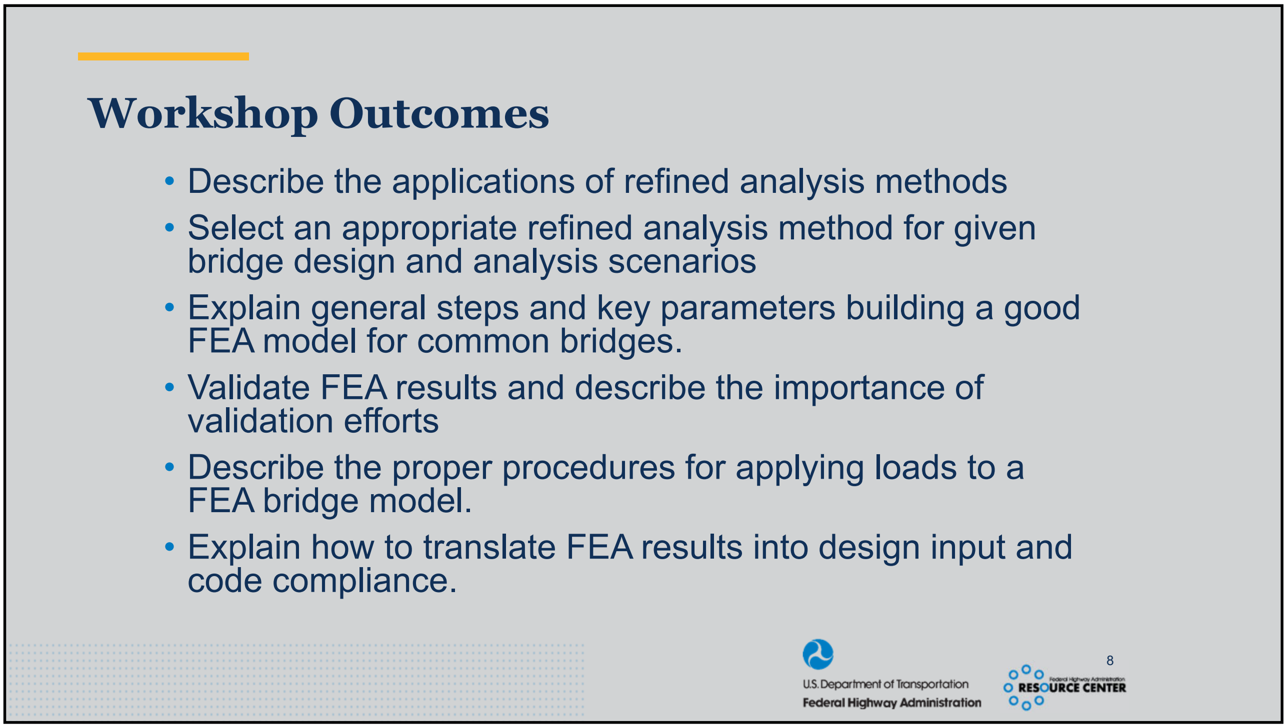
Which finite element analysis (FEA) software package you have the most experience with?

a) SAP 2000
b) MDX
c) MIDAS Civil
d) LUSAS
e) LARSA
f) ANSYS
g) ABAQUS
h) Others
i) None



Workshop Outline

Lesson 0 - Workshop Introduction
Lesson 1 - Fundamentals of Finite Element Analysis (FEA) & Modeling
Lesson 2 - General Girder Bridge Modeling
Lesson 3 - Modeling Special Topics
Lesson 4 - Load Applications
Lesson 5 - Problem set-up and analysis procedures using P/S girder bridge example
Lesson 6 - Verification/Validation of results
Lesson 7 - Analysis to Design
Closeout – Assessment and Workshop Evaluations



Workshop Outcomes

- Describe the applications of refined analysis methods
- Select an appropriate refined analysis method for given bridge design and analysis scenarios
- Explain general steps and key parameters building a good FEA model for common bridges.
- Validate FEA results and describe the importance of validation efforts
- Describe the proper procedures for applying loads to a FEA bridge model.
- Explain how to translate FEA results into design input and code compliance.

Lesson 0. Introduction

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Learning Outcomes

- Describe the difference between conventional analysis vs refined analysis.
- Explain why refined analysis is needed or its benefits.

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Introduction

Background

- Computation mechanics and software enable engineers moving away from girder-by-girder approximate procedures to a system analysis approach.
- In contrary, current U.S. specifications and practice still rely heavily on simplified, approximate analyses to determine the structural effects of vehicle loading on bridge girders.
- Obstacles include the lack of software, lack of training, lack of specifications, complexity and perceived high cost-to-benefit ratio.
- Benefits of line girder analysis – simple, uniform safety, adequate and conservative for square slab and slab & girder structures

Introduction

Background (cont'd)

- European scan on "Assuring Bridge Safety & Serviceability (ABSS) , June 2009:
  - Five countries: Finland, Austria, Germany, France & UK.
  - Two main focus areas:
    - Safety & serviceability measures during design, construction, and operation.
    - Refined analysis applications from design thru operation.

Introduction

Background (cont’d)

- Important Findings on “Refined Analysis” from the scan:
  - No line girder distribution factors in design codes – Refined analysis predominates analysis, design, assessment of existing bridges.
  - As minimum, use 2-D elastic analysis (grillage). Approximate methods for quick review of refined analysis.
  - Increased model sophistication provides more accurate and most likely, higher available load carrying capacity.
- FHWA response to NTSB recommendation as a result of FIU pedestrian bridge collapse investigation emphasized on the importance of verifying results generated from refined analysis .

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Introduction

Conventional Analyses

- 1-D, line girder analyses
- Distribution factors to account for live load distribution
- Long history, easily checked, simple concepts
- Does not work well for curved and highly skewed bridges
- Reasonably conservative estimates demands on girders
- Diaphragms and cross frame forces are not calculated
- Works well for ‘non-skewed’ and multi-girder bridges

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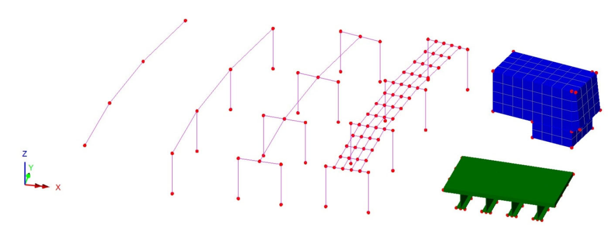
Refined Analyses

- Directly models transverse behavior of bridge
- Plate with Eccentric Beam (PEB), shell elements used for deck
- Grid or Grillage
- Shell element webs, beam element flanges
- Include diaphragms and cross frames
- Full 3D modeling of girders and deck
- Solid, volume, brick elements



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Analysis Dimensions

- Defining by dimensions of analysis can lead to confusion
  - 3D model can use 1D elements
  - 1D model can use 3D elements
- Definition adopted based on dimensions required to fully define results
  - 1D: results are dependent on just one coordinate, e.g. beam line analysis
  - 2D: results are dependent on two coordinates, e.g. grillage or PEB.
  - 3D: results are dependent on three coordinates

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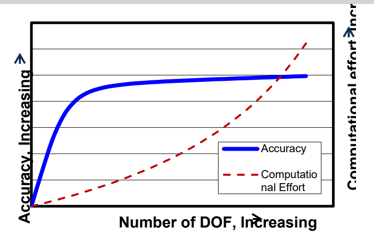
Introduction

Why refined analysis?

- ***Required* by AASHTO LRFD** - where the specification approximate methods do not apply.
- **Better understanding of behavior** - by more rigorous assessment of limit states.
- **Increased economy** - by going beyond use of approximate, conservative design formulae.
- **Improved safety evaluation-** by full consideration of condition data such as section losses or as-built geometry
- **Increased sustainability** - by more frequent salvaging of existing infrastructure
- **Accelerated innovation development** - as industry gains deeper understanding of bridge behavior
- **NCHRP 12-62** suggests 10% - 30% (slab on steel I girders) & 10% - 20% (slab on concrete I girders) live load reduction compared to LRFD LL Distribution factors.

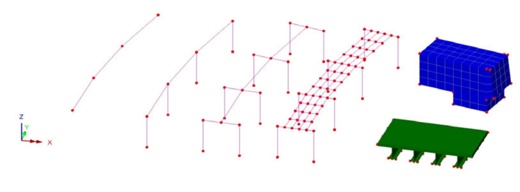


When to Use Refined Analysis

- Multi-girder bridges when:
  - Curved
  - Skews exceed 30 degrees
  - Complex geometry
  - Significant savings possible
- Complex, long span structures
  - Cable stayed, arch, etc.
- Redundancy evaluations
  - Investigate effects of loss of members
  - New guidance from AASHTO
- Complex details
  - Local stress concentration analysis (shear lag)

State of Practice (Order of complexity)

- Line girder analysis (1D) with the use of D.F.*
- Grillage Analogy (2D) & PEB methods*
- Orthotropic plate theory methods
- Folded plate theory methods
- Semi-continuum methods
- Finite strip methods*
- Pseudo 3D finite element methods*
- Full 3D finite element methods*



Learning Outcome Review

- Describe the difference between conventional analysis vs refined analysis.
- Explain why refined analysis is needed or its benefits.

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Questions?

Next:
Lesson 1 Fundamental Finite Element Method & Modeling (Part 1)

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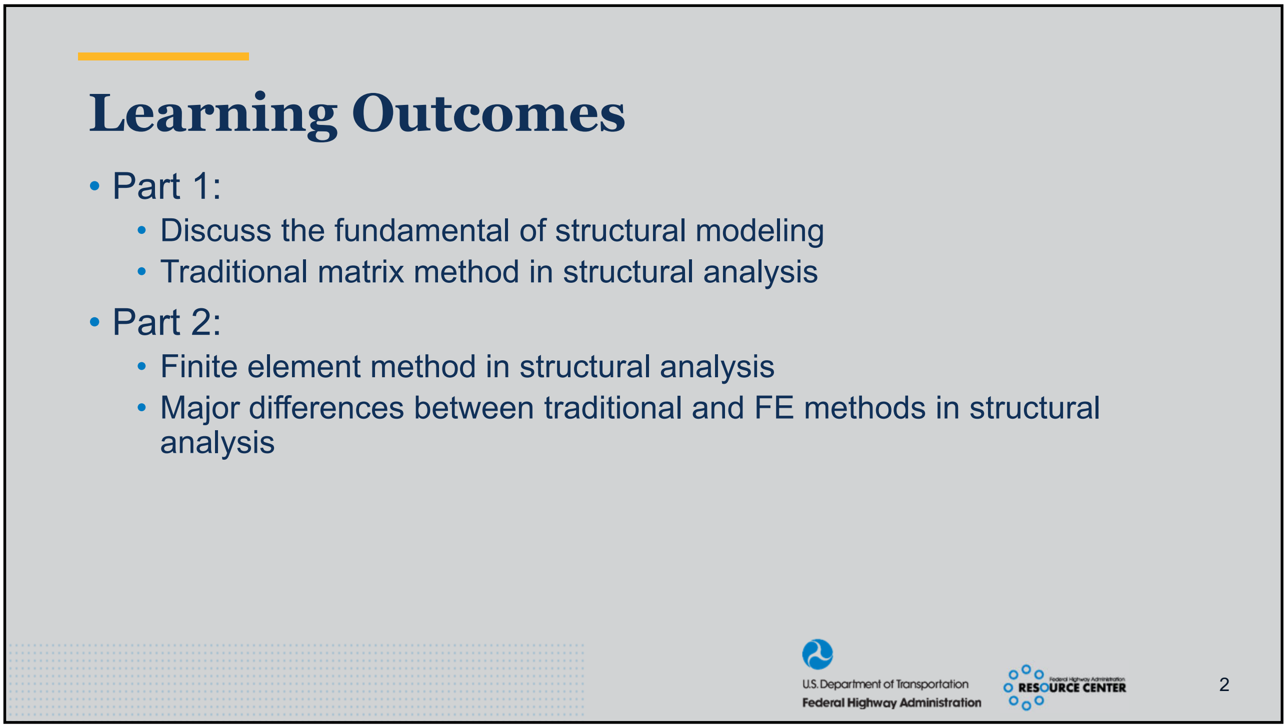
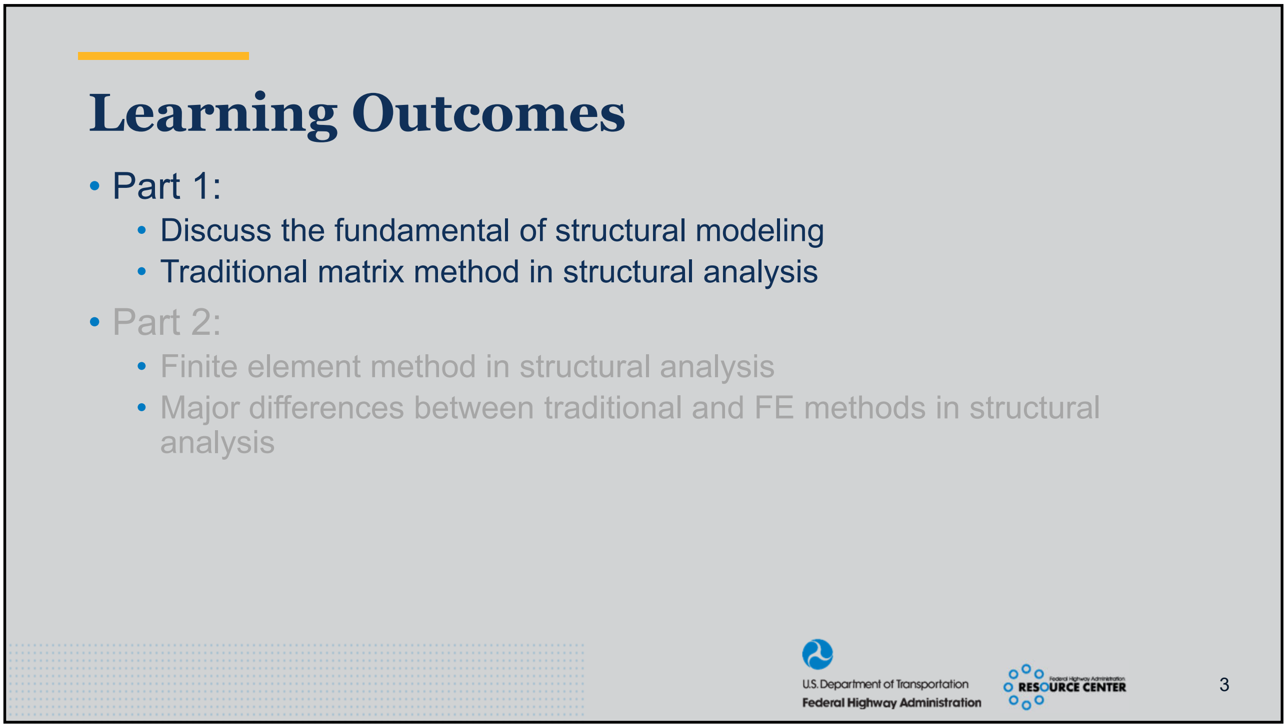
Lesson 1 Fundamental Finite Element Method & Modeling (Part 1)



Learning Outcomes

- Part 1:
  - Discuss the fundamental of structural modeling
  - Traditional matrix method in structural analysis
- Part 2:
  - Finite element method in structural analysis
  - Major differences between traditional and FE methods in structural analysis



Learning Outcomes

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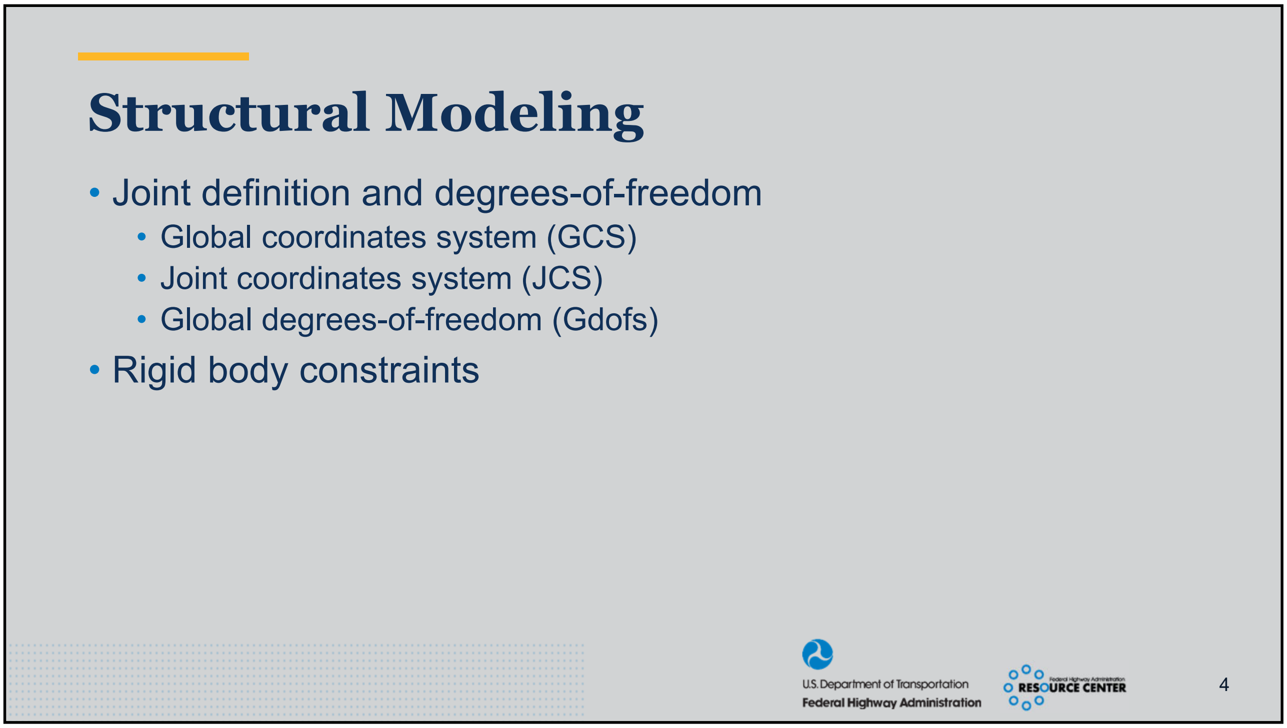
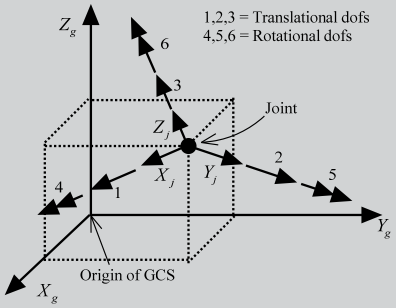


Structural Modeling

- Joint definition and degrees-of-freedom
  - Global coordinates system (GCS)
  - Joint coordinates system (JCS)
  - Global degrees-of-freedom (Gdofs)
- Rigid body constraints

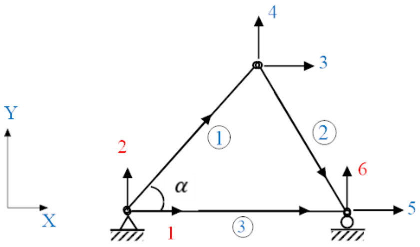
Joint definition and Degrees-of-Freedom (FHWA Manual 2.3.3)

- Joint
  - The point where two or more elements are connected
- Global coordinates system (GCS)
  - GCS, (X_g, Y_g, Z_g), defines the locations of structural joints
- Joint coordinates system (JCS)
  - JCS, (X_j, Y_j, Z_j), defines the directions of a joint's global degrees-of-freedom (Gdofs)
  - Joint Gdofs: 3 translational & 3 rotational



Joint definition and Degrees-of-Freedom

- Gdofs
  - Restrained dof
  - Free dof
  - Rigid body constrained dof
  - Condensed dof



- Total # of global dofs = 6
- Free dofs = 3,4,5
- Restrained dofs = 1, 2, 6

Rigid body constrain (FHWA Manual 1.4)

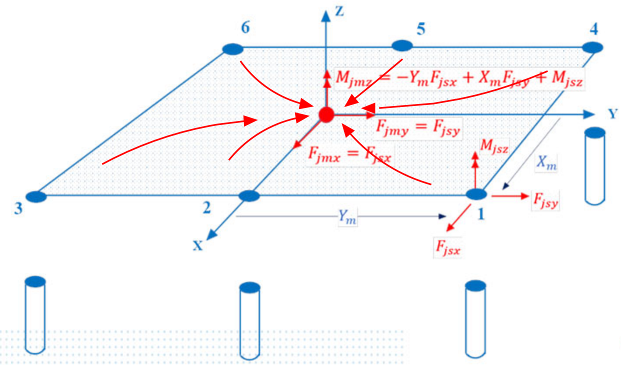
- Rigid body constrained dofs
  - If relative deformation between two joints is very small, two joints can be considered as being constrained by a rigid body.
  - The movement of one joint (the “slave” joint) can be determined by the movement of the other joint (the “master” joint)
  - The dofs for the slave joint can be ignored
  - The total # of Gdofs can be reduced.

2-D Plane rigid body constrained dofs

- In-plane – rigid
- Out-of-plane - flexible

2-D Plane rigid body constrained dofs

- Assume slave joints at column locations (18 dofs)
- Transfer column forces from slave joints to master joint through $[T_m]$
- Reduce 18 dofs at slave joints to 3 dofs at master joint



Slave joint

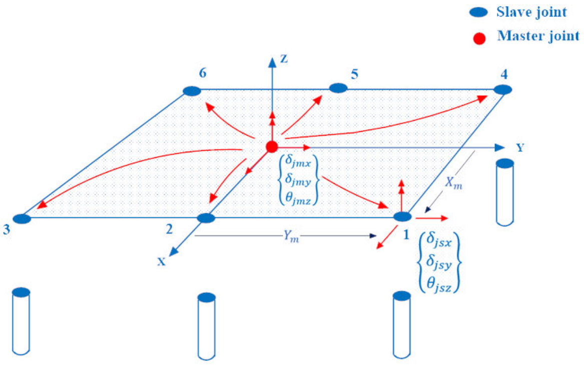
Master joint

$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Y_m & X_m & 1 \end{bmatrix} \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ M_{jsz} \end{Bmatrix} = [T_m] \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ M_{jsz} \end{Bmatrix}$$

Transformation matrix

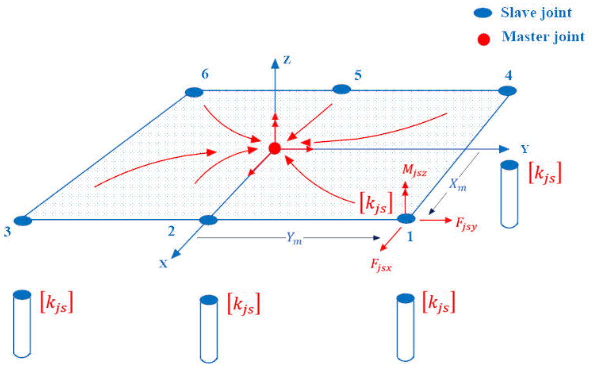
2-D Plane rigid body constrained dofs

- Transfer displacements at master joint back to slave joints through $[T_m]^T$
- $$\begin{Bmatrix} \delta_{jsx} \\ \delta_{jsy} \\ \theta_{jsz} \end{Bmatrix} = [T_m]^T \begin{Bmatrix} \delta_{jmx} \\ \delta_{jmy} \\ \theta_{jmz} \end{Bmatrix}$$
- See appendix A

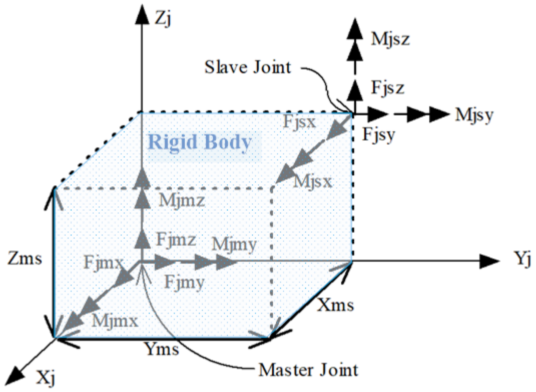


2-D Plane rigid body constrained dofs

- Transfer column stiffness matrix to master joint through:
- $\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ M_{jmz} \end{Bmatrix} = \sum [T_m][k_{js}][T_m]^T \begin{Bmatrix} \delta_{jmx} \\ \delta_{jmy} \\ \theta_{jmz} \end{Bmatrix}$
- $= [k_{jm}] \begin{Bmatrix} \delta_{jmx} \\ \delta_{jmy} \\ \theta_{jmz} \end{Bmatrix}$
- See Appendix A for derivation of $[k_{jm}]$



3-D Rigid body constrained dofs (Appendix A)



$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ F_{jmz} \\ M_{jmx} \\ M_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -Z_{ms} & Y_{ms} & 1 & 0 & 0 \\ Z_{ms} & 0 & -X_{ms} & 0 & 1 & 0 \\ -Y_{ms} & X_{ms} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ F_{jsz} \\ M_{jsx} \\ M_{jsy} \\ M_{jsz} \end{Bmatrix}$$

or

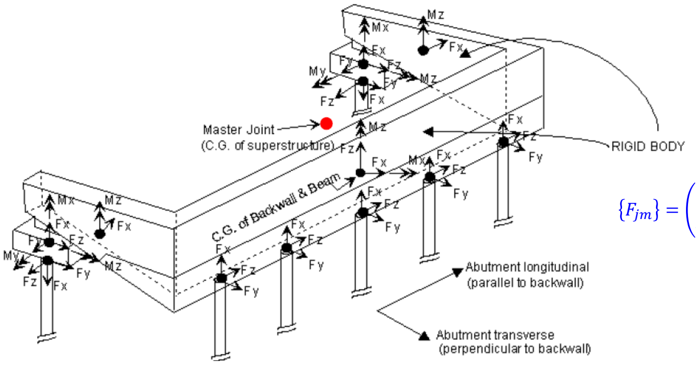
$$\{F_{jm}\} = [T_{ms}]\{F_{js}\}$$

Similarly,

$$\{\delta_{js}\} = [T_{ms}]^T \{\delta_{jm}\}$$

3-D Rigid body constrained dofs

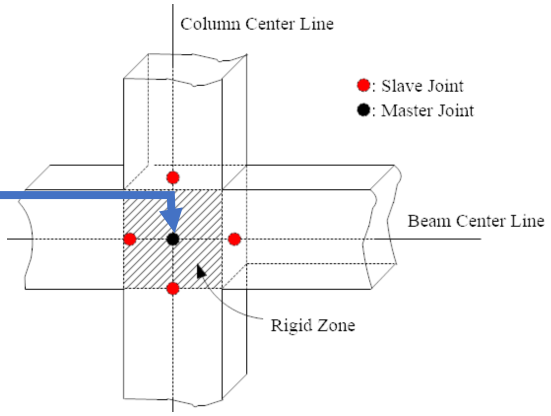
- Rigid body transformation



$$\{F_{jm}\} = \left(\sum [T_m][K_{js}][T_m]^T\right)\{\delta_{jm}\} \quad = [K_{jm}]\{\delta_{jm}\}$$

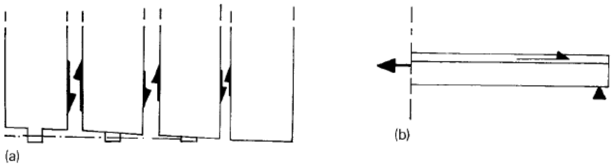
Application of rigid body constrain

- Region zones
- $\{F_{jm}\} = [T_m][K_{js}][T_m]^T\{\delta_{jm}\}$
- $= [K_{jm}]\{\delta_{jm}\}$
- $[K_m] = \sum_{i=1}^{4}[T_m]_i[K_{js}]_i[T_m]_i^T$

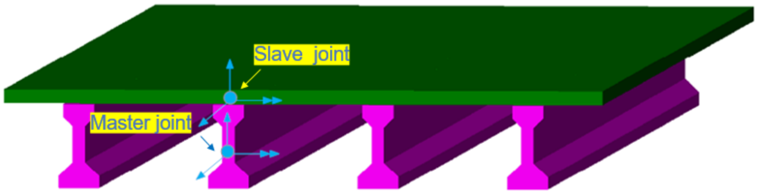


Application of rigid body constrain

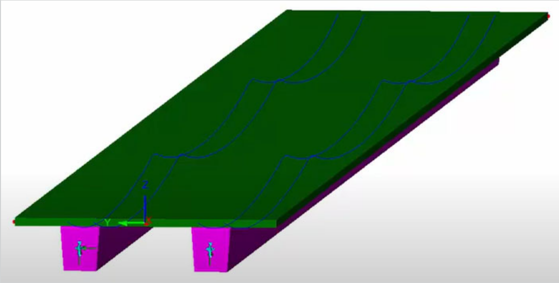
- Traditional line or grillage analysis:

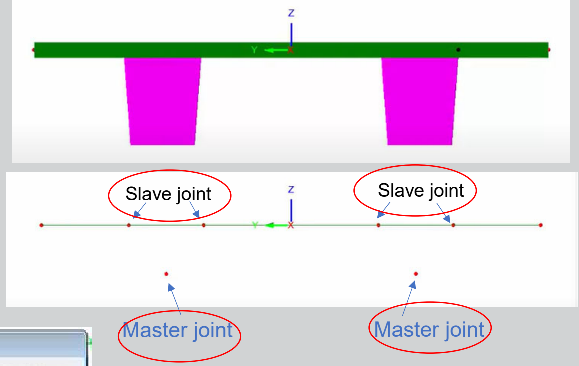


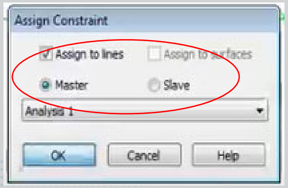
- Plate with eccentric beams:



Application of rigid body constrain







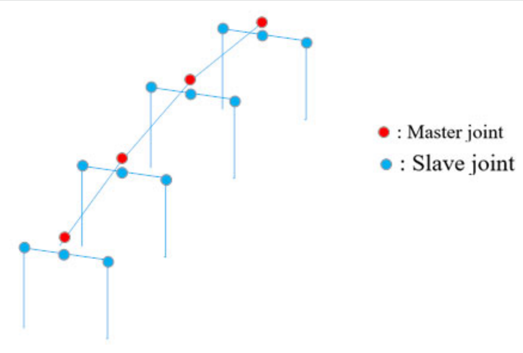
Assign Constraint

☑ Assign to lines ☐ Assign to surfaces

● Master ○ Slave

Analysis 1

Application of rigid body constrain



(spine model)

Knowledge Review -- Question 1(L1A S18 Q1)

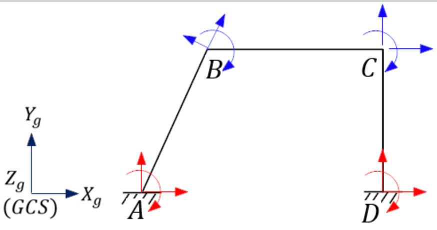
For the structure shown below, which of the following statements are true? Select all that apply.

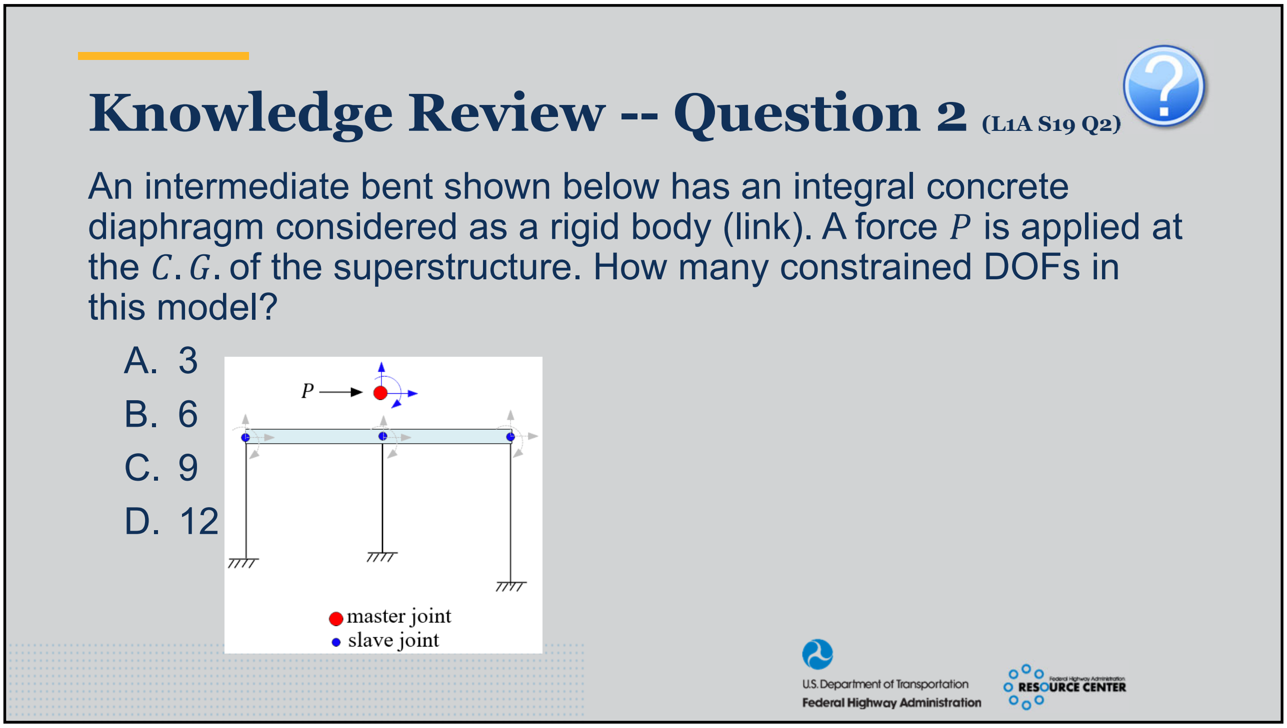
A. The JCS at joint B is not coincident with GCS

B. The JCS at joint C is coincident with GCS

C. Total # of free DOFs is 6

D. All of the above





Knowledge Review -- Question 2 (L1A S19 Q2)

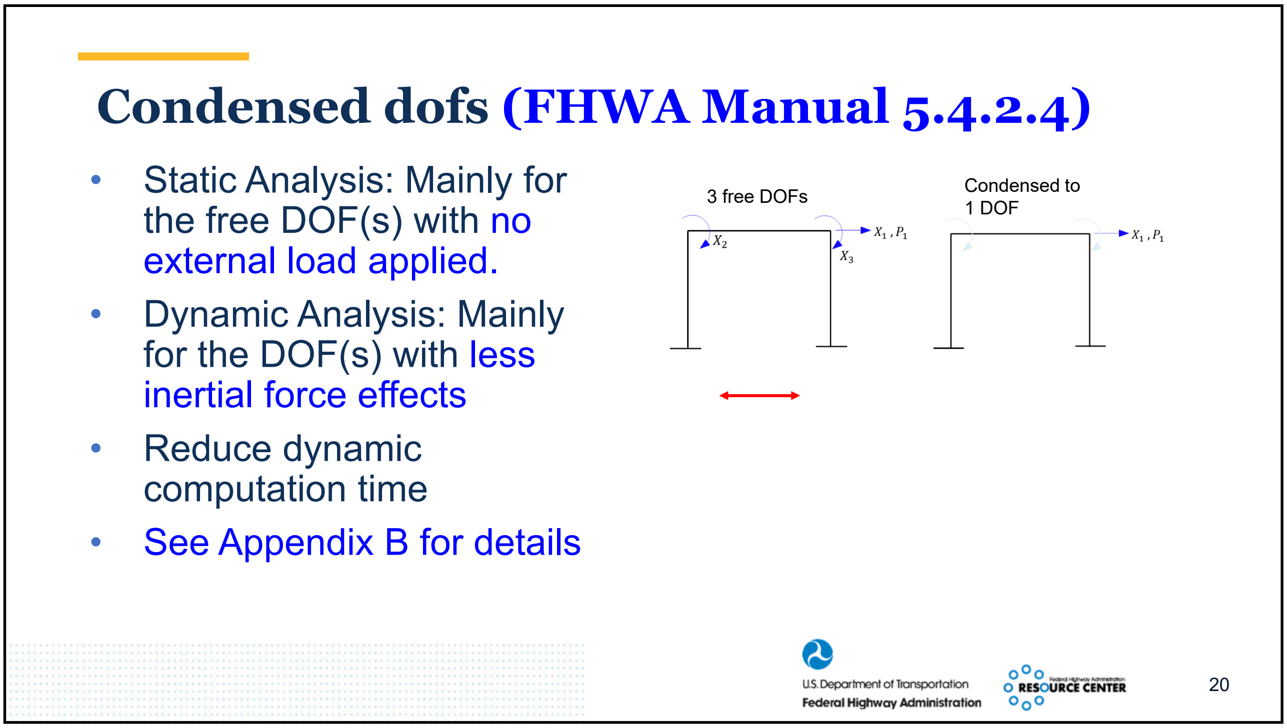
An intermediate bent shown below has an integral concrete diaphragm considered as a rigid body (link). A force P is applied at the $C.G.$ of the superstructure. How many constrained DOFs in this model?

A. 3

B. 6

C. 9

D. 12



Condensed dofs (FHWA Manual 5.4.2.4)

- Static Analysis: Mainly for the free DOF(s) with no external load applied.
- Dynamic Analysis: Mainly for the DOF(s) with less inertial force effects
- Reduce dynamic computation time
- See Appendix B for details

3 free DOFs

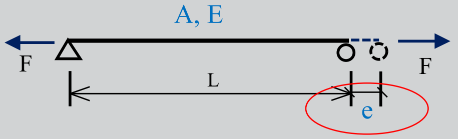
Condensed to 1 DOF

Traditional Element Stiffness Matrix Formulation (FHWA Manual 2.3.4)

- Element coordinate system (ECS)
- Truss element
- Beam element
- Spring element
- Point element

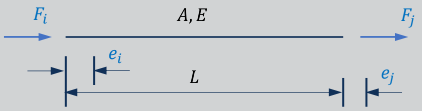
Traditional Element Stiffness Formulation

- Truss element (1dof)
  - $F = ke = \frac{AE}{L}e$
- Truss element (2 dofs)
  - $F_i = \frac{AE}{L}(e_i - e_j)$
  - $F_j = \frac{AE}{L}(e_j - e_i)$
  - $\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} e_i \\ e_j \end{Bmatrix}$



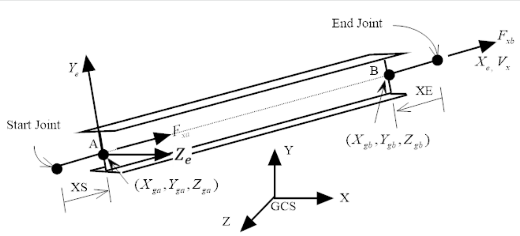
Traditional Element Stiffness Formulation

- Truss element (2 dofs)
  - ECS: (X_e, Y_e, Z_e)
  - ECS may not coincide with GCS
  - Can have rigid zone (eccentricity) at each end



$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & \frac{-AE}{L} \\ \frac{-AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} e_i \\ e_j \end{Bmatrix}$$

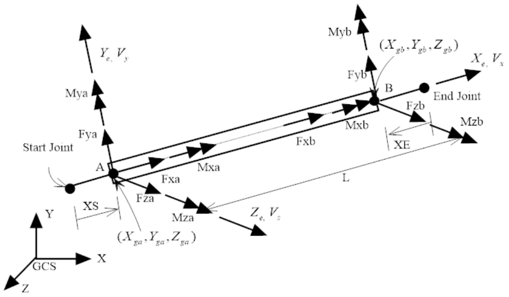
$$= \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{Bmatrix} e_i \\ e_j \end{Bmatrix}$$



k_{ij} = Move one unit displacement at dof j direction, what the force required in dof i direction is.

Traditional Element Stiffness Formulation

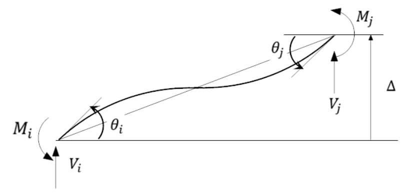
- Beam element (maximum of 12 dofs)
  - ECS (X_e, Y_e, Z_e) may not coincide with GCS
  - Can have rigid zone (eccentricity) at each end



$$\{F_e\} = \begin{Bmatrix} F_{xa} \\ F_{ya} \\ F_{za} \\ M_{xa} \\ M_{ya} \\ M_{za} \\ F_{xb} \\ F_{yb} \\ F_{zb} \\ M_{xb} \\ M_{yb} \\ M_{zb} \end{Bmatrix} \quad \{\delta_e\} = \begin{Bmatrix} \delta_{xa} \\ \delta_{ya} \\ \delta_{za} \\ \theta_{xa} \\ \theta_{ya} \\ \theta_{za} \\ \delta_{xb} \\ \delta_{yb} \\ \delta_{zb} \\ \theta_{xb} \\ \theta_{yb} \\ \theta_{zb} \end{Bmatrix}$$

Traditional Element Stiffness Formulation

- Beam bending & shear stiffnesses
  - Based on classical slope-deflection method



$$M_i = \frac{2EI}{l}(2\theta_i + \theta_j - 3\frac{\Delta}{l})$$

$$M_j = \frac{2EI}{l}(\theta_i + 2\theta_j - 3\frac{\Delta}{l})$$

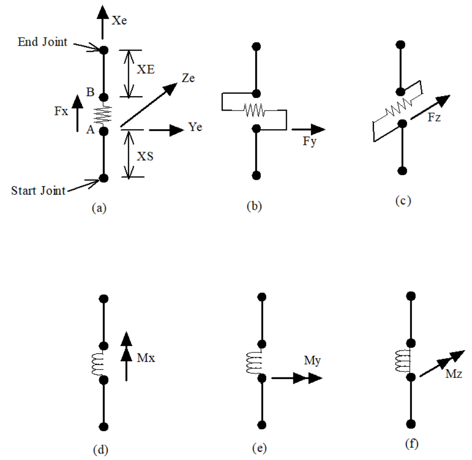
$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

From the equilibrium condition of M_i, M_j, V_i, V_j

$$\begin{Bmatrix} M_i \\ M_j \\ V_i \\ V_j \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{-12EI}{L^3} \\ \frac{-6EI}{L^2} & \frac{-6EI}{L^2} & \frac{-12EI}{L^3} & \frac{12EI}{L^3} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \\ \Delta_i \\ \Delta_j \end{Bmatrix}$$

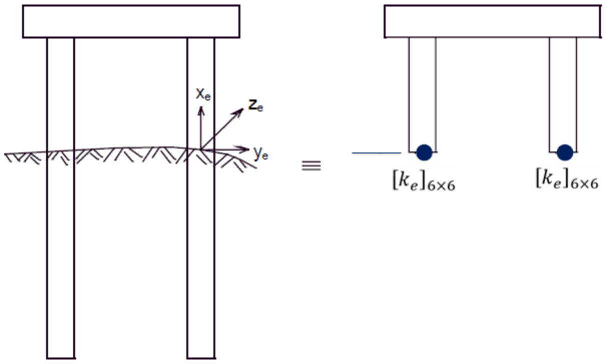
Traditional Element Stiffness Formulation

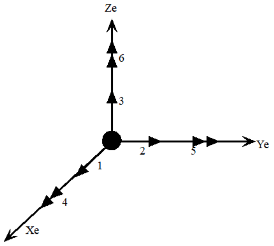
- Spring element (1dof)
  - May be oriented in one of 6 positions
  - Distance between the start & end joint can be assigned as zero.



Traditional Element Stiffness Formulation

- Point element (6 dofs)
  - A $6x6$ stiffness matrix
  - Bridge foundation stiffnesses can be modeled by point elements.



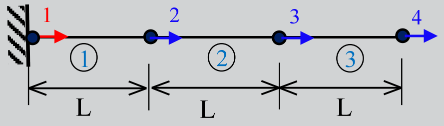


Formulation of structural global stiffness matrix

- By Direct Element (Stiffness) method (FHWA Manual 1.4)
  - Define GCS & Gdofs (both free and restrained)
  - Formulate each member stiffness matrix corresponding to element local dofs in its ECS direction
  - Convert each ECS element stiffness matrix to the Gdof directions
  - Stack up all element stiffness matrix in the Gdof directions to form structural global stiffness matrix

Direct Element Method (FHWA Manual 1.4)

- Define global DOFs (both free and restrained)
- Formulate each member stiffness matrix in its ECS direction



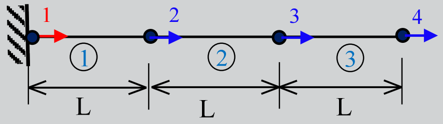
$$\begin{Bmatrix}F_i\\F_j\end{Bmatrix} = \begin{bmatrix}\frac{AE}{L} & -\frac{AE}{L}\\ -\frac{AE}{L} & \frac{AE}{L}\end{bmatrix}\begin{Bmatrix}e_i\\e_j\end{Bmatrix}$$

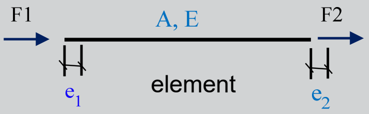
1 = restrained global DOFs

2,3,4 = free global DOFs

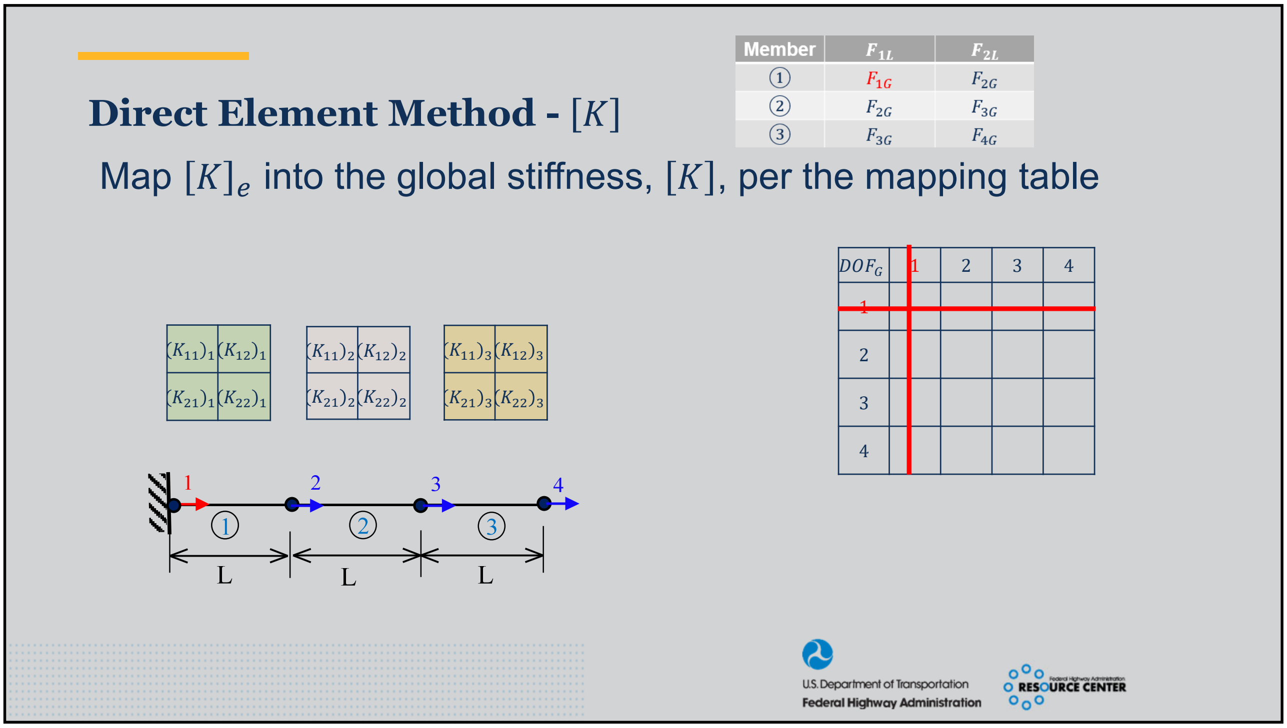
Direct Element Method – Mapping Table

Develop mapping to define associations between local and global dofs





Member	F_{1L}	F_{2L}
①	F_{1G}	F_{2G}
②	F_{2G}	F_{3G}
③	F_{3G}	F_{4G}



Direct Element Method - $[K]$

Member	F_{1L}	F_{2L}
①	F_{1G}	F_{2G}
②	F_{2G}	F_{3G}
③	F_{3G}	F_{4G}

Map $[K]_e$ into the global stiffness, $[K]$, per the mapping table

$(K_{11})_1$	$(K_{12})_1$
$(K_{21})_1$	$(K_{22})_1$

$(K_{11})_2$	$(K_{12})_2$
$(K_{21})_2$	$(K_{22})_2$

$(K_{11})_3$	$(K_{12})_3$
$(K_{21})_3$	$(K_{22})_3$

DOF_G	1	2	3	4
1				
2				
3				
4				

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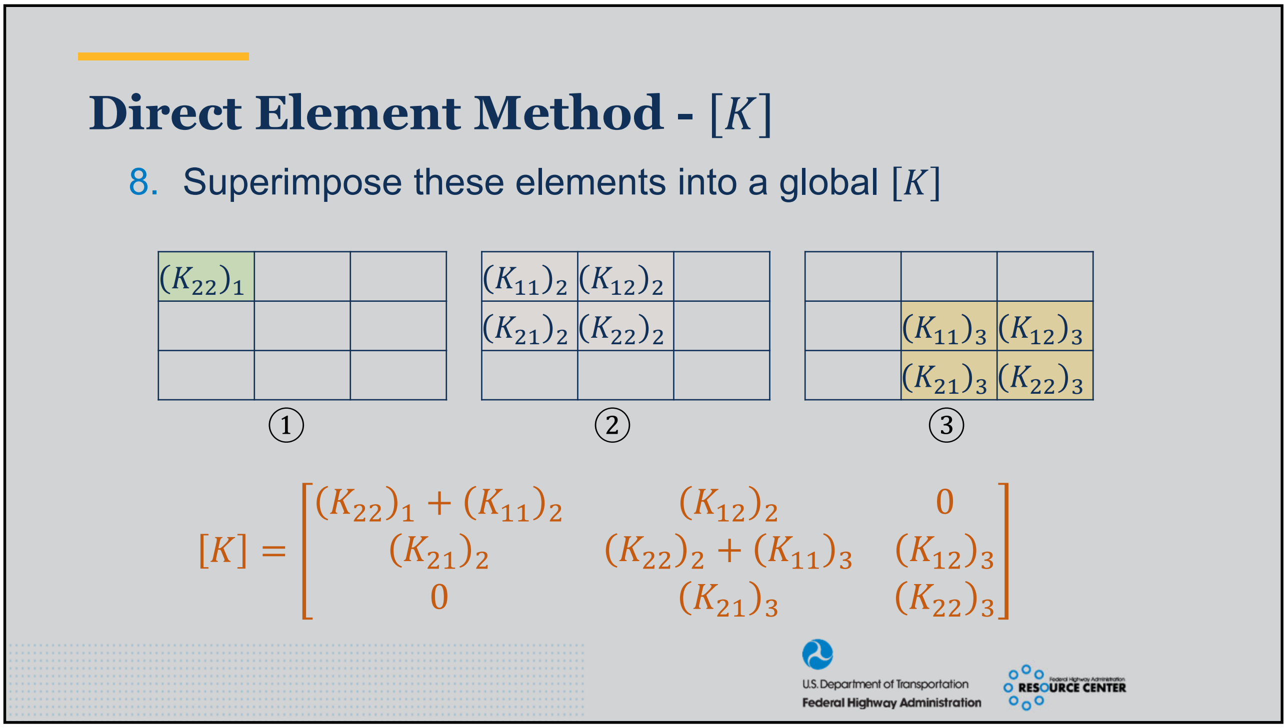

Direct Element Method - $[K]$

8. Superimpose these elements into a global $[K]$

$(K_{22})_1$		

①

$(K_{11})_2$	$(K_{12})_2$	
$(K_{21})_2$	$(K_{22})_2$	

②

	$(K_{11})_3$	$(K_{12})_3$
	$(K_{21})_3$	$(K_{22})_3$

③

$$[K] = \begin{bmatrix} (K_{22})_1 + (K_{11})_2 & (K_{12})_2 & 0 \\ (K_{21})_2 & (K_{22})_2 + (K_{11})_3 & (K_{12})_3 \\ 0 & (K_{21})_3 & (K_{22})_3 \end{bmatrix}$$

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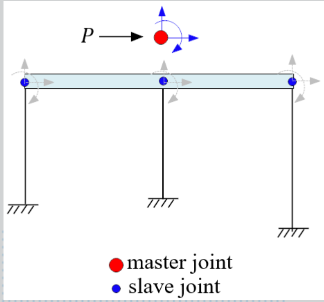
Knowledge Review -- Question 3 (L1A S33 Q3)

An intermediate bent shown below has an integral concrete diaphragm considered as a rigid body (link). A force P is applied at the $C.G.$ of the superstructure. What's the size of the structural stiffness matrix corresponding free DOFs?

A. 3 x 3
B. 9 x 9
C. 12 x 12
D. 21 x 21



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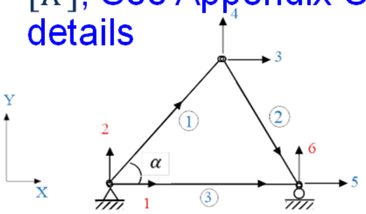
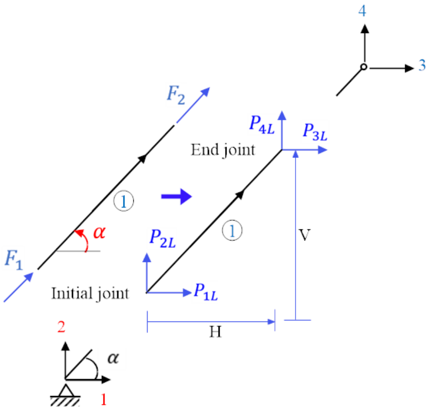
Direct Element Method

- Convert element $[S]_e$ in its ECS to the Gdof direction
    - Force equilibrium
- Map $[K]_e$ into the global stiffness, $[K]$, See Appendix C for more details



1,2,6 = restrained global DOFs
3,4,5 = free global DOFs



$$\{P_L\} = [A]_e\{F_e\}$$

(See appendix C)

$$\{P_L\} = ([A]_e[S]_e[A]_e^T)\{X_L\}$$

$$= [K]_e\{X_L\}$$

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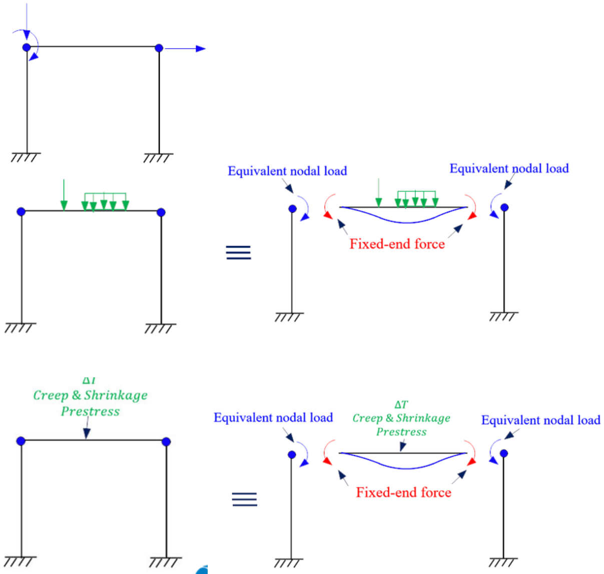
Modeling Loads

Joint load:

- Load applied at a joint

Element load:

- External element load
- Internal element load
    - Temperature, ΔT
    - Creep and shrinkage
    - Prestress, support settlement
- Transferred to the joints (i.e. Equivalent nodal loads)
    - Through axial ε, M-ϕ w/ conjugate beam theory
    - Get fixed-end forces



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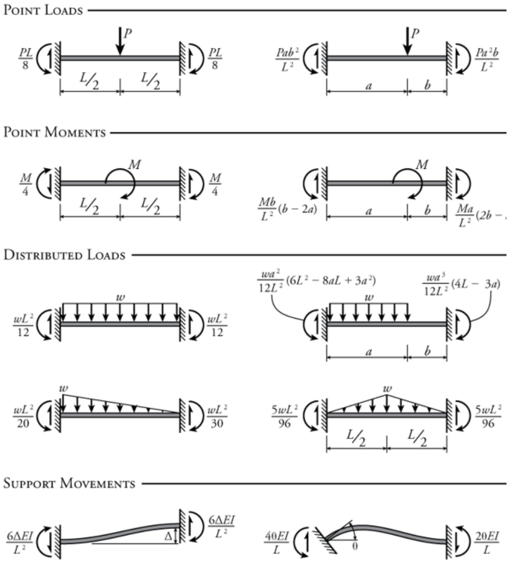
Modeling Loads

- Typical fixed-end moments due to external element loads



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Traditional matrix method in structural analysis

- Advantages:
  - Based on conventional mechanics of materials
  - Close-formed element stiffness matrix formulation
  - Member nodal forces & deformations are calculated directly
- Disadvantages:
  - Member stress & strain can not be calculated directly
  - The close-formed stiffness matrices for 2-D (plate & shell) and 3-D (solid) elements are very difficult to derived, based on conventional mechanics of materials.

Traditional matrix method in structural analysis

- Disadvantages (cont.):
  - Can not handle stress concentration problems and capture the 2-D or 3-D stress field.
  - Composite action between two elements (i.e. deck & girder) can not be modeled adequately.

Learning Outcomes Review

- Part 1:
  - Overview of structural modeling
  - Traditional matrix method in structural analysis
  - Advantage/Disadvantage of traditional matrix method

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Next:
Lesson 1: Fundamental Finite Element Method & Modeling (Part 2)

Appendix A

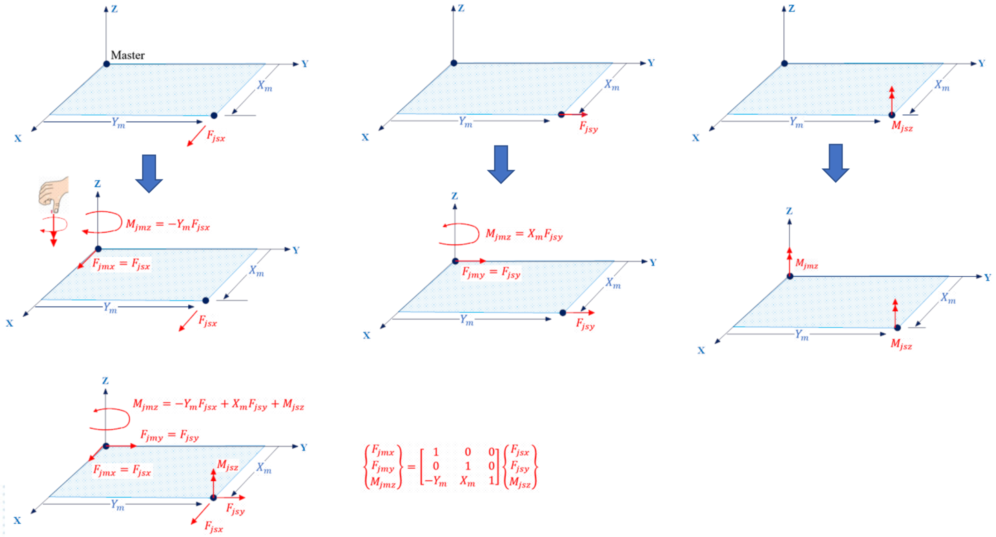
- 2-D Rigid Body Transformation
- 3-D Rigid Body Transformation

Rigid body constrain (FHWA Manual 1.4)

- Rigid body constrained dofs
  - If relative deformation between two joints is very small, two joints can be considered as being constrained by a rigid body.
  - The movement of one joint (the “slave” joint) can be determined by the movement of the other joint (the “master” joint)
  - The dofs for the slave joint can be ignored
  - The total # of Gdofs can be reduced.

X-Y Rigid Plane Example

Rigid - in plane;
Flexible - out of plane



$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Y_m & X_m & 1 \end{bmatrix} \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ M_{jsz} \end{Bmatrix}$$

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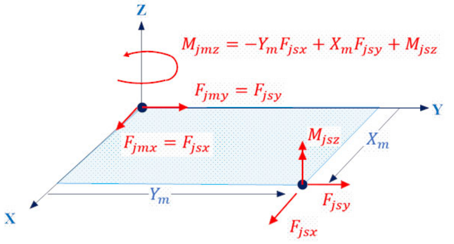
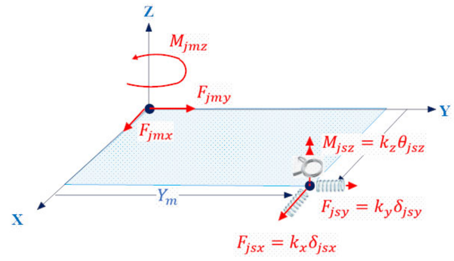
Example:

X-Y rigid plane



$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Y_m & X_m & 1 \end{bmatrix} \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ M_{jsz} \end{Bmatrix}$$

$[T_m]$



$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Y_m & X_m & 1 \end{bmatrix} \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{Bmatrix} \delta_{jsx} \\ \delta_{jsy} \\ \theta_{jsz} \end{Bmatrix}$$

$$= [T_m][k_{js}] \begin{Bmatrix} \delta_{jsx} \\ \delta_{jsy} \\ \theta_{jsz} \end{Bmatrix} \qquad (1)$$

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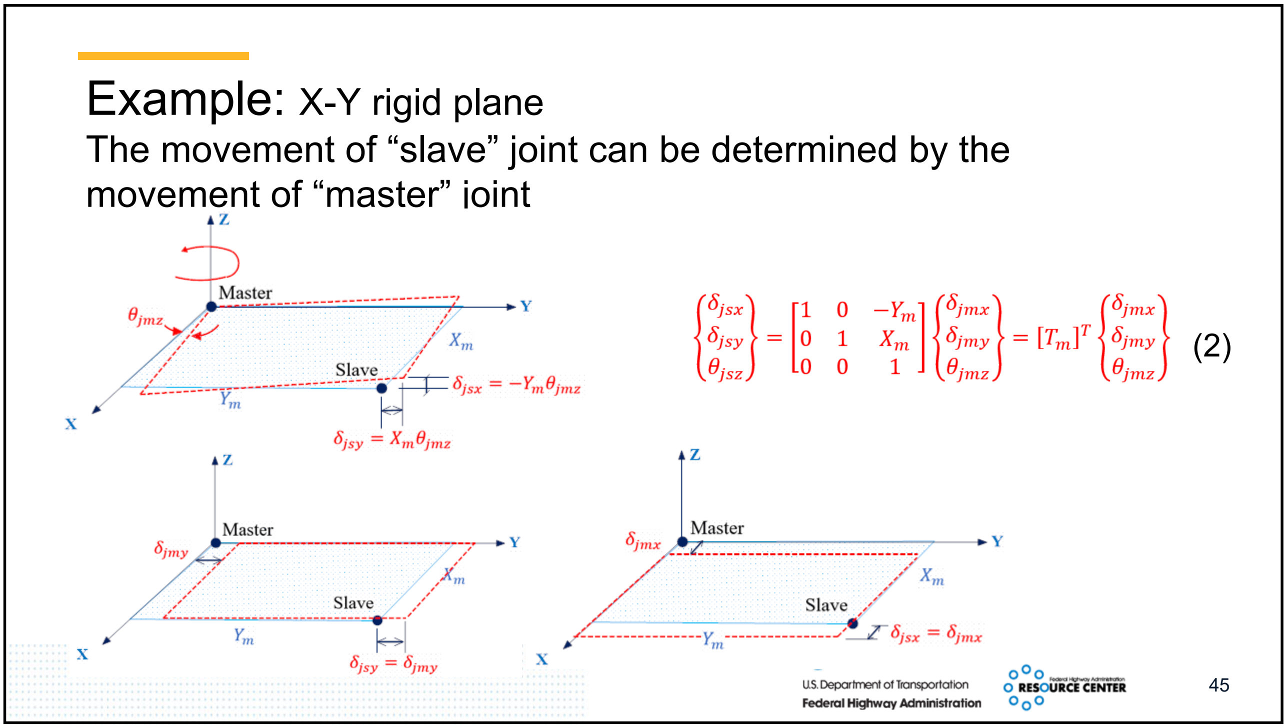

Example: X-Y rigid plane

The movement of “slave” joint can be determined by the movement of “master” joint

$$\begin{Bmatrix}\delta_{jsx}\\ \delta_{jsy}\\ \theta_{jsz}\end{Bmatrix} = \begin{bmatrix}1 & 0 & -Y_m\\ 0 & 1 & X_m\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\delta_{jmx}\\ \delta_{jmy}\\ \theta_{jmz}\end{Bmatrix} = [T_m]^T\begin{Bmatrix}\delta_{jmx}\\ \delta_{jmy}\\ \theta_{jmz}\end{Bmatrix} \quad (2)$$

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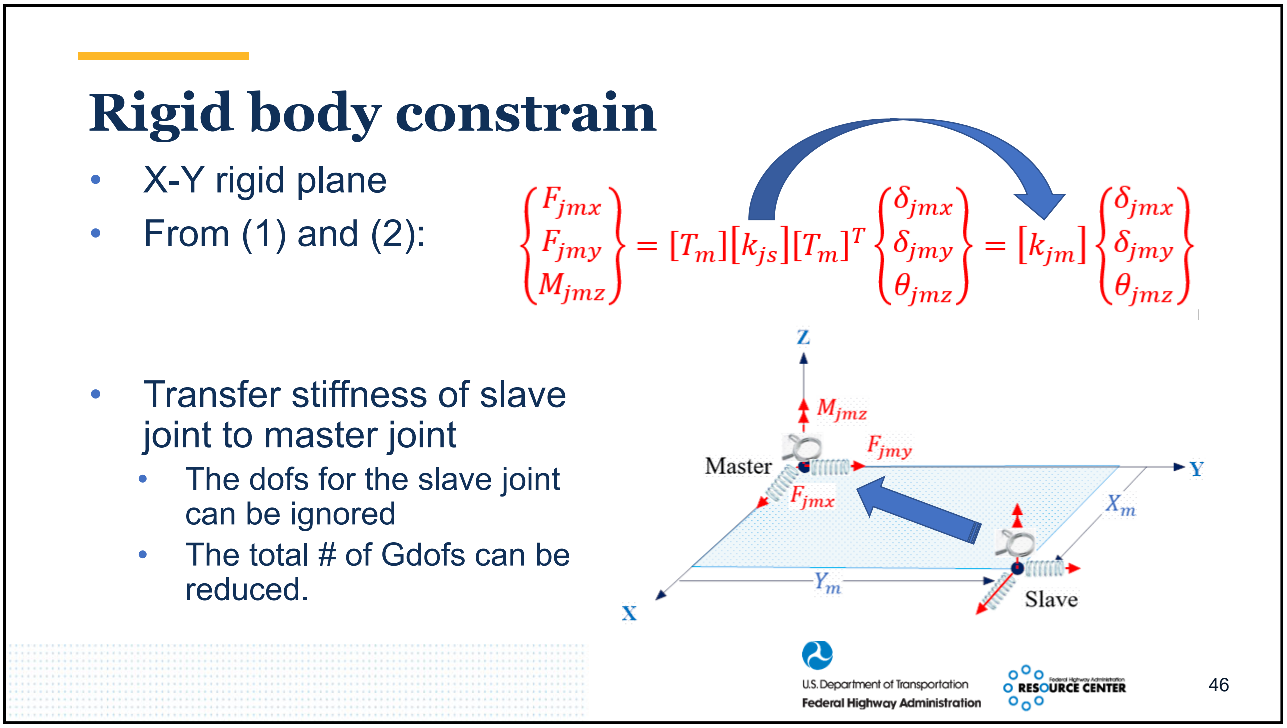

Rigid body constrain

- X-Y rigid plane
- From (1) and (2):

$$\begin{Bmatrix}F_{jmx}\\ F_{jmy}\\ M_{jmz}\end{Bmatrix} = [T_m][k_{js}][T_m]^T\begin{Bmatrix}\delta_{jmx}\\ \delta_{jmy}\\ \theta_{jmz}\end{Bmatrix} = [k_{jm}]\begin{Bmatrix}\delta_{jmx}\\ \delta_{jmy}\\ \theta_{jmz}\end{Bmatrix}$$

- Transfer stiffness of slave joint to master joint
  - The dofs for the slave joint can be ignored
  - The total # of Gdofs can be reduced.

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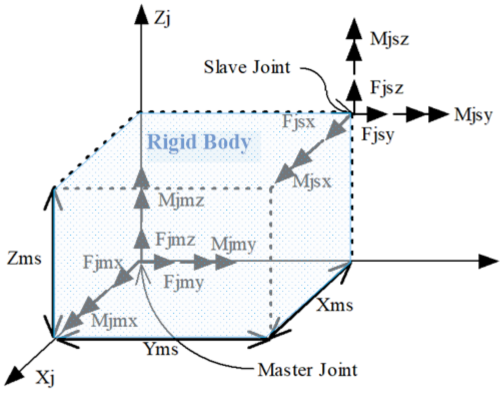
Rigid body constrain

- 3-D rigid body constraints



$$\begin{Bmatrix} F_{jmx} \\ F_{jmy} \\ F_{jmz} \\ M_{jmx} \\ M_{jmy} \\ M_{jmz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -Z_{ms} & Y_{ms} & 1 & 0 & 0 \\ Z_{ms} & 0 & -X_{ms} & 0 & 1 & 0 \\ -Y_{ms} & X_{ms} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{jsx} \\ F_{jsy} \\ F_{jsz} \\ M_{jsx} \\ M_{jsy} \\ M_{jsz} \end{Bmatrix}$$

or

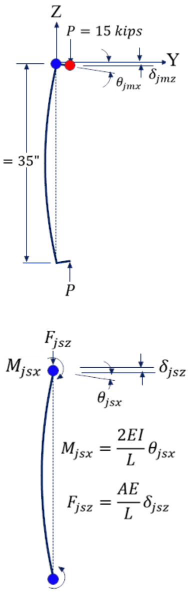
$$\{F_{jm}\} = [T_{ms}]\{F_{js}\}$$

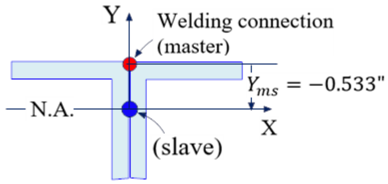
Similarly,

$$\{\delta_{js}\} = [T_{ms}]^T\{\delta_{jm}\}$$

Example 1:

- A double-angle member in the truss diaphragm is subjected to 15 kips axial load at connection point. Find the axial deformation and end rotation of the member at its centroid.





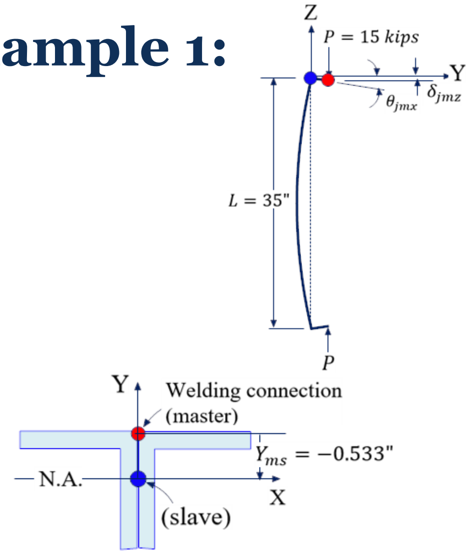
$$A = 1.88\ in^2;\ I_{xx} = 0.695\ in^4$$

$$[k_{js}] = \begin{bmatrix} \frac{AE}{L} & 0 \\ 0 & \frac{2EI}{l} \end{bmatrix} = \begin{bmatrix} 1558 & 0 \\ 0 & 1151.7 \end{bmatrix}$$

$$\{F_{jm}\} = \begin{Bmatrix} F_{jmz} \\ M_{jmx} \end{Bmatrix} = [T_m]\begin{Bmatrix} F_{jsz} \\ M_{jsx} \end{Bmatrix}$$

$$[T_m] = \begin{bmatrix} 1 & 0 \\ Y_{ms} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.533 & 1 \end{bmatrix}$$

Example 1:



$$[k_{jm}] = [T_m][k_{js}][T_m]^T$$
$$= \begin{bmatrix} 1558 & -830.414 \\ -830.414 & 1594 \end{bmatrix}$$

$$\{P\} = \begin{Bmatrix} P_{jmz} \\ M_{jmx} \end{Bmatrix} = \begin{Bmatrix} -15\ kips \\ 0\ k-in \end{Bmatrix}$$

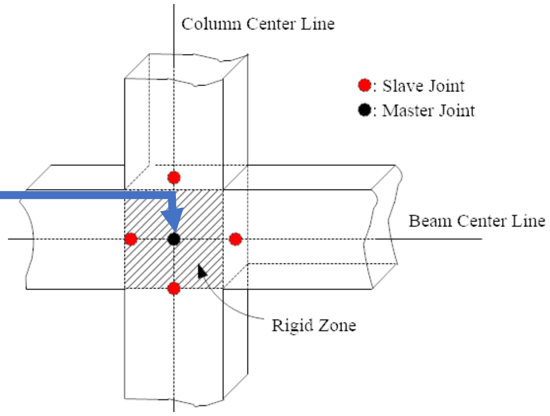
Since $[k_{jm}] \begin{Bmatrix} \delta_{jmz} \\ \theta_{jmx} \end{Bmatrix} = \begin{Bmatrix} P_{jmz} \\ M_{jmx} \end{Bmatrix}$

$$\therefore \begin{Bmatrix} \delta_{jmz} \\ \theta_{jmx} \end{Bmatrix} = [k_{jm}]^{-1} \begin{Bmatrix} P_{jmz} \\ M_{jmx} \end{Bmatrix} = \begin{Bmatrix} -0.0133\ in \\ -0.0069\ rad \end{Bmatrix}$$

$$\begin{Bmatrix} \delta_{jsz} \\ \theta_{jsx} \end{Bmatrix} = [T_m]^T \begin{Bmatrix} \delta_{jmz} \\ \theta_{jmx} \end{Bmatrix} = \begin{Bmatrix} -0.0096\ in \\ -0.0069\ rad \end{Bmatrix}$$

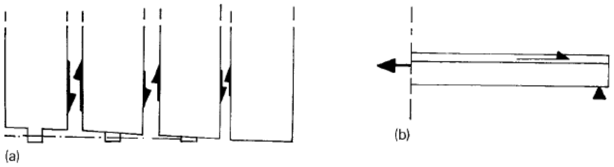
Application of rigid body constrain

- Region zones
- $\{F_{jm}\} = [T_m][K_{js}][T_m]^T\{\delta_{jm}\}$
- $= [K_{jm}]\{\delta_{jm}\}$
- $[K_m] = \sum_{i=1}^{4}[T_m]_i[K_{js}]_i[T_m]_i^T$

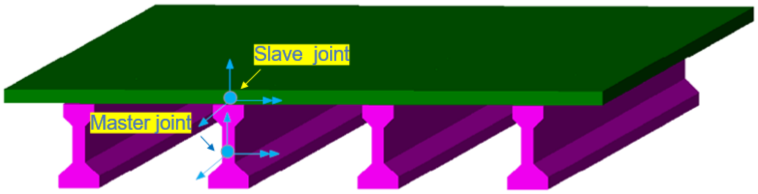


Application of rigid body constrain

- Traditional line or grillage analysis:



- Plate with eccentric beams:



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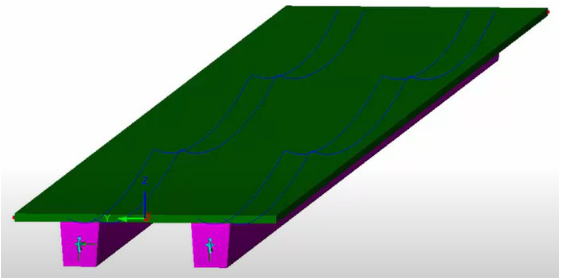
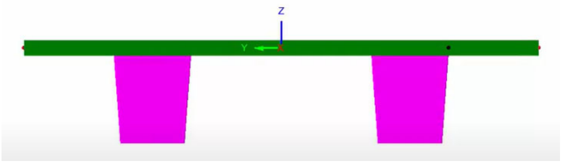
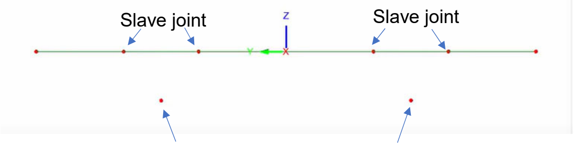
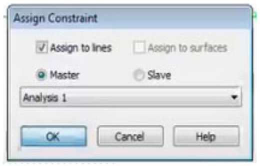
Application of rigid body constrain








Assign Constraint

☑ Assign to lines ☐ Assign to surfaces

● Master ○ Slave

Analysis 1

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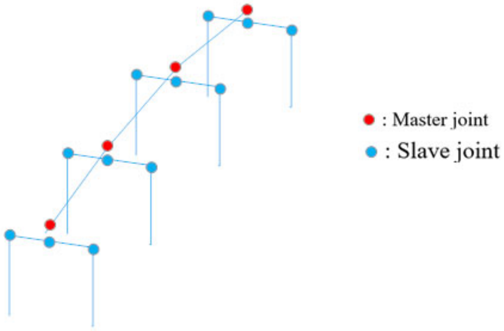
Application of rigid body constrain



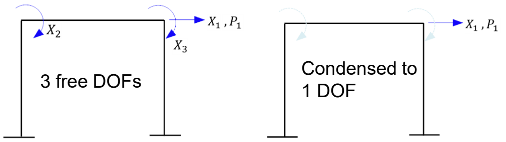
(spine model)

Appendix B

- Condensed Dofs (FHWA Manual 5.4.2.4)

Joint definition and Degrees-of-Freedom

- Condensed dofs (FHWA Manual 5.4.2.4)
  - Static Analysis: Mainly for the DOF(s) without applied external load.
  - Dynamic Analysis: Mainly for the DOF(s) with less inertial force effects



$$\left[\begin{array}{c|cc} k_{11} & k_{12} & k_{13} \\ \hline k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{array}\right] \left\{\begin{array}{c} X_1 \\ \hline X_2 \\ X_3 \end{array}\right\} = \left\{\begin{array}{c} P_1 \\ \hline 0 \\ 0 \end{array}\right\}$$

$$\left[\begin{array}{c|c} [k_{ff}] & [k_{fc}] \\ \hline [k_{cf}] & [k_{cc}] \end{array}\right] \left\{\begin{array}{c} X_f \\ \hline X_c \end{array}\right\} = \left\{\begin{array}{c} P_f \\ \hline 0 \end{array}\right\}$$

$$[k_{ff}]\{X_f\} + [k_{fc}]\{X_c\} = \{P_f\}$$

$$\therefore \{X_c\} = -[k_{cc}]^{-1}[k_{cf}]\{X_f\} \quad (1)$$

$$[k_{cf}]\{X_f\} + [k_{cc}]\{X_c\} = 0 \quad (2)$$

Substitute (1) into (2)

$$\underbrace{-([k_{fc}][k_{cc}]^{-1}[k_{cf}] + [k_{fc}])}_{[k]_c}\{X_f\} = \{P_f\} \Rightarrow [k]_c\{X_f\} = \{P_f\}$$

Joint definition and Degrees-of-Freedom

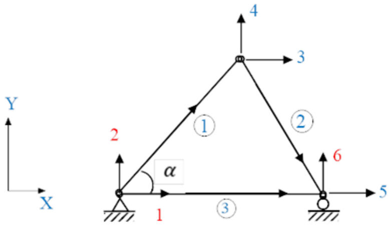
- Condensed dofs
  - Mainly for the dofs with less inertial force effects in the dynamic analysis.
  - To partition the original free dofs into condensed dofs and remaining free dofs
  - Reduce dynamic computation time
  - Not much benefit for the static analysis

Appendix C

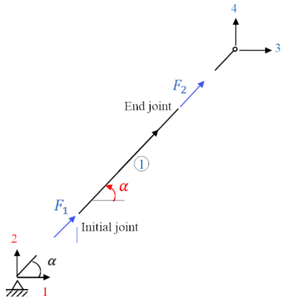
- Direct Element Method
- Numerical Example

Direct Element Method

- Define global DOFs (both free and restrained)
- Formulate each member stiffness matrix in its ECS direction



1,2,6 = restrained global DOFs
3,4,5 = free global DOFs

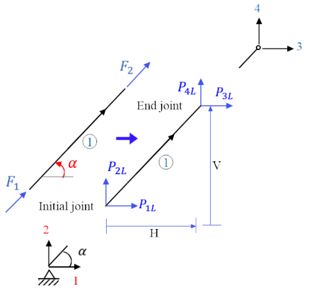


$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \begin{bmatrix}\frac{AE}{L} & \frac{-AE}{L}\\ \frac{-AE}{L} & \frac{AE}{L}\end{bmatrix}\begin{Bmatrix}e_1\\e_2\end{Bmatrix} = [S]_{e(2x2)}\begin{Bmatrix}e_1\\e_2\end{Bmatrix}$$

$$\{F_e\} = [S]_e\{\delta_e\}$$

Direct Element Method

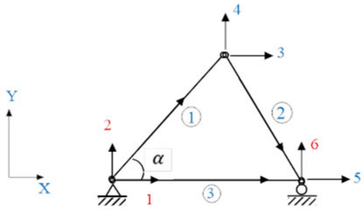
- Convert element $[S]_e$ in its ECS to the Gdof direction
  - Force equilibrium



$$P_{1L} = F_1 \cos\alpha$$
$$P_{2L} = F_1 \sin\alpha$$
$$P_{3L} = F_2 \cos\alpha$$
$$P_{4L} = F_2 \sin\alpha$$

$$\therefore \begin{Bmatrix} P_{1L} \\ P_{2L} \\ P_{3L} \\ P_{4L} \end{Bmatrix} = \begin{bmatrix} cos\alpha & 0 \\ sin\alpha & 0 \\ 0 & cos\alpha \\ 0 & sin\alpha \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = [A]_e \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\{P_L\} = [A]_e\{F_e\}$$



1,2,6 = restrained global DOFs

3,4,5 = free global DOFs

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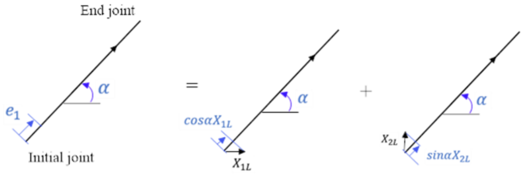
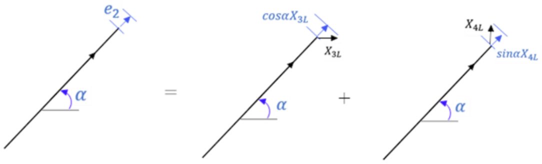
Direct Element Method

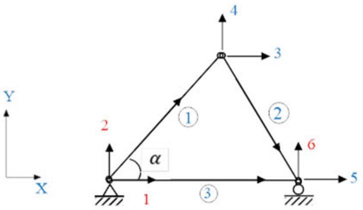
- Displacement Compatibility



$$e_1 = cos\alpha X_{1L} + sin\alpha X_{2L}$$



$$e_2 = cos\alpha X_{3L} + sin\alpha X_{4L}$$



$$\therefore \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha & 0 & 0 \\ 0 & 0 & cos\alpha & sin\alpha \end{bmatrix} \begin{Bmatrix} X_{1L} \\ X_{2L} \\ X_{3L} \\ X_{4L} \end{Bmatrix}$$

$$\{\delta_e\} = [A]_e{}^T\{X_L\}$$

$$\{P_L\} = [A]_e\{F_e\}$$
$$= [A]_e([S]_e\{\delta_e\})$$
$$= [A]_e[S]_e([A]_e{}^T\{X_L\})$$
$$\therefore \{P_L\} = [K]_e\{X_L\}$$
$$[K]_{e(4x4)} = [A]_{e(4x2)}[S]_{e(2x2)}[A]^T_{e(2x4)}$$

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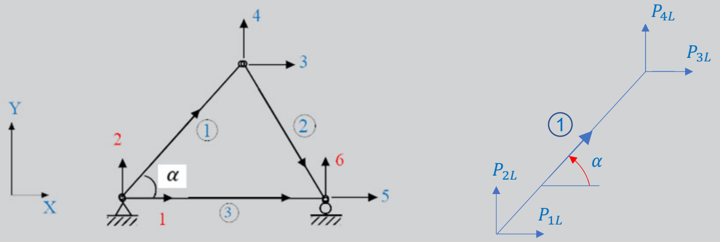
Direct Element Method – Mapping Table

Develop mapping to define associations between local and global dofs

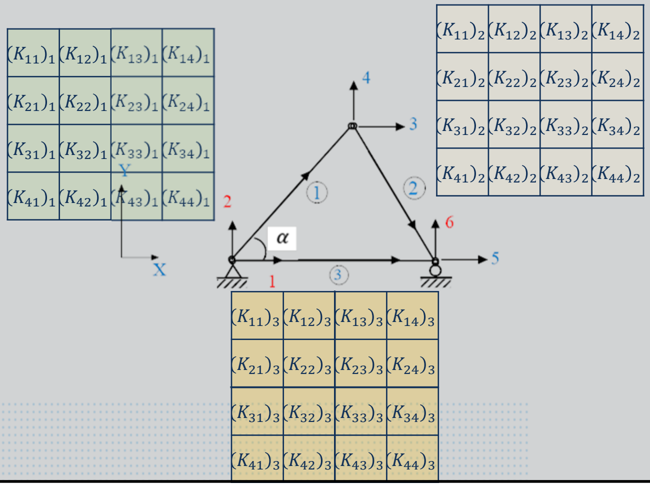


Member	P_{1L}	P_{2L}	P_{3L}	P_{4L}
①	P_{1G}	P_{2G}	P_{3G}	P_{4G}
②	P_{3G}	P_{4G}	P_{5G}	P_{6G}
③	P_{1G}	P_{2G}	P_{5G}	P_{6G}

Direct Element Method - $[K]$

Member	P_{1L}	P_{2L}	P_{3L}	P_{4L}
①	P_{1G}	P_{2G}	P_{3G}	P_{4G}
②	P_{3G}	P_{4G}	P_{5G}	P_{6G}
③	P_{1G}	P_{2G}	P_{5G}	P_{6G}

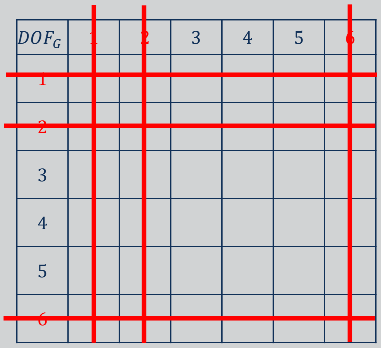
Map $[K]_e$ into the global stiffness, $[K]$, per the mapping table



$(K_{11})_1$	$(K_{12})_1$	$(K_{13})_1$	$(K_{14})_1$
$(K_{21})_1$	$(K_{22})_1$	$(K_{23})_1$	$(K_{24})_1$
$(K_{31})_1$	$(K_{32})_1$	$(K_{33})_1$	$(K_{34})_1$
$(K_{41})_1$	$(K_{42})_1$	$(K_{43})_1$	$(K_{44})_1$

$(K_{11})_2$	$(K_{12})_2$	$(K_{13})_2$	$(K_{14})_2$
$(K_{21})_2$	$(K_{22})_2$	$(K_{23})_2$	$(K_{24})_2$
$(K_{31})_2$	$(K_{32})_2$	$(K_{33})_2$	$(K_{34})_2$
$(K_{41})_2$	$(K_{42})_2$	$(K_{43})_2$	$(K_{44})_2$

$(K_{11})_3$	$(K_{12})_3$	$(K_{13})_3$	$(K_{14})_3$
$(K_{21})_3$	$(K_{22})_3$	$(K_{23})_3$	$(K_{24})_3$
$(K_{31})_3$	$(K_{32})_3$	$(K_{33})_3$	$(K_{34})_3$
$(K_{41})_3$	$(K_{42})_3$	$(K_{43})_3$	$(K_{44})_3$



DOF_G	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Direct Element Method - $[K]$

8. Superimpose these elements into a global $[K]$

$(K_{33})_1$	$(K_{34})_1$	
$(K_{43})_1$	$(K_{44})_1$	

①

$(K_{11})_2$	$(K_{12})_2$	$(K_{13})_2$
$(K_{21})_2$	$(K_{22})_2$	$(K_{23})_2$
$(K_{31})_2$	$(K_{32})_2$	$(K_{33})_2$

②

		$(K_{33})_3$

③

$$[K] = \begin{bmatrix} (K_{33})_1 + (K_{11})_2 & (K_{34})_1 + (K_{12})_2 & (K_{13})_2 \\ (K_{43})_1 + (K_{21})_2 & (K_{44})_1 + (K_{22})_2 & (K_{23})_2 \\ (K_{31})_2 & (K_{32})_2 & (K_{33})_2 + (K_{33})_3 \end{bmatrix}$$

Direct Element Method - Analysis

Use $[K]$ to calculate $\{X\}, \{F\}_e$, and $\{e\}_e$

$$\{X\} = [K]^{-1}\{P\}$$

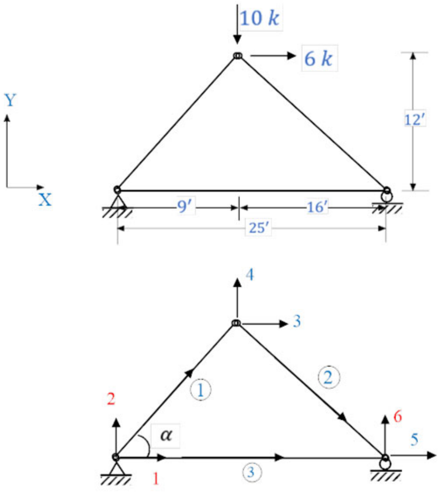
$$\{e\}_e = [A]_e{}^T\{X\}_{P_{1G}-P_{4G}}$$

$$\{F\}_e = [s]_e\{e\}_e$$

Where: $\{X\}_{P_{1G}-P_{4G}}$ are the global displacements corresponding to the element's $P_{1L} - P_{4L}$

Example 2:

- Use direct element method
  - Formulate structural $[K]$
  - Find member 1 internal forces and deformations

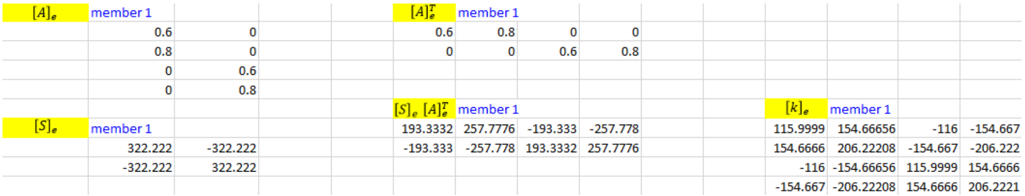


Example 2:

- Convert member stiffness, $[s]_e$, to joint global directions, $[K]_e$
  - Member 1:

$$L = \sqrt{9^2 + 12^2} = 15' = 180“$$

$$sin\alpha = \frac{V}{L} = \frac{12}{15} = 0.8; cos\alpha = \frac{H}{L} = \frac{9}{15} = 0.6$$



$[A]_e$	member 1	
	0.6	0
	0.8	0
	0	0.6
	0	0.8

$[S]_e$	member 1	
	322.222	-322.222
	-322.222	322.222

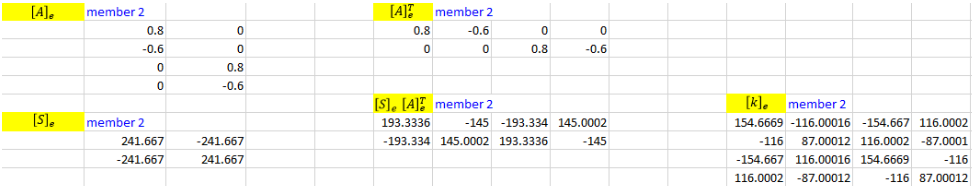
$[A]_e^T$	member 1		
0.6	0.8	0	0
0	0	0.6	0.8

$[S]_e$ $[A]_e^T$	member 1		
193.3332	257.7776	-193.333	-257.778
-193.333	-257.778	193.3332	257.7776

$[k]_e$	member 1		
115.9999	154.66656	-116	-154.667
154.6666	206.22208	-154.667	-206.222
-116	-154.66656	115.9999	154.6666
-154.667	-206.22208	154.6666	206.2221

Example 2:

- Member 2:



$[A]_e$	member 2
0.8	0
-0.6	0
0	0.8
0	-0.6

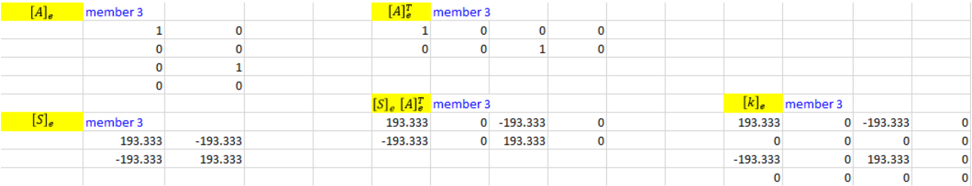
$[A]_e^T$	member 2		
0.8	-0.6	0	0
0	0	0.8	-0.6

$[S]_e$	member 2
241.667	-241.667
-241.667	241.667

$[S]_e\ [A]_e^T$	member 2		
193.3336	-145	-193.334	145.0002
-193.334	145.0002	193.3336	-145

$[k]_e$	member 2		
154.6669	-116.00016	-154.667	116.0002
-116	87.00012	116.0002	-87.0001
-154.667	116.00016	154.6669	-116
116.0002	-87.00012	-116	87.00012

- Member 3:



$[A]_e$	member 3
1	0
0	0
0	1
0	0

$[A]_e^T$	member 3		
1	0	0	0
0	0	1	0

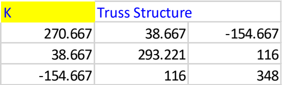
$[S]_e$	member 3
193.333	-193.333
-193.333	193.333

$[S]_e\ [A]_e^T$	member 3		
193.333	0	-193.333	0
-193.333	0	193.333	0

$[k]_e$	member 3		
193.333	0	-193.333	0
0	0	0	0
-193.333	0	193.333	0
0	0	0	0

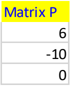
Example 2:

- Formulate structural $[K]$ by mapping $[K]_e$ to Gdofs 3,4 & 5

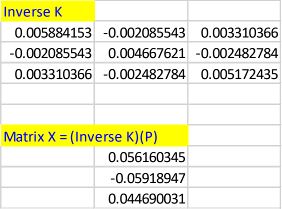


K	Truss Structure	
270.667	38.667	-154.667
38.667	293.221	116
-154.667	116	348

- Apply global force vector, $\{P\}$:
- Find global displacement vector $\{X\}$:



Matrix P
6
-10
0

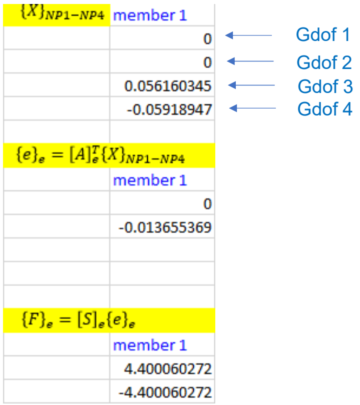


Inverse K		
0.005884153	-0.002085543	0.003310366
-0.002085543	0.004667621	-0.002482784
0.003310366	-0.002482784	0.005172435

Matrix X = (Inverse K)(P)	
	0.056160345
	-0.05918947
	0.044690031

Example 2:

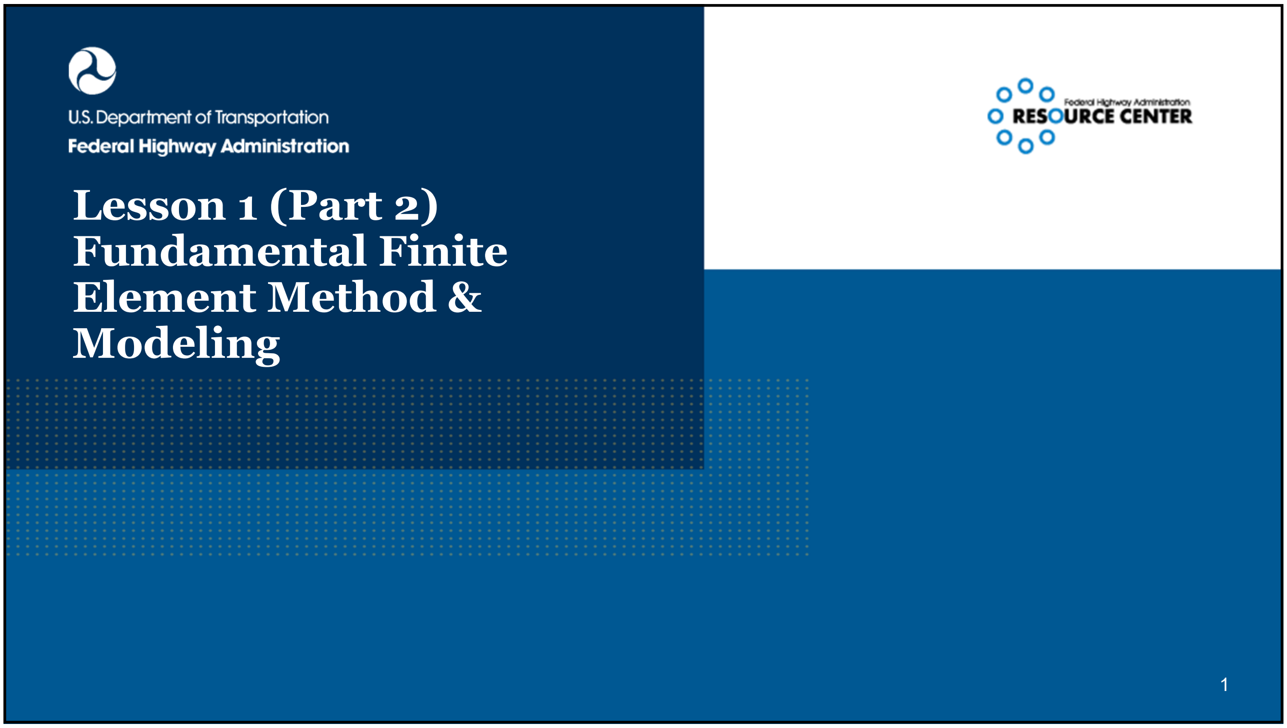
- Find member 1 internal forces and deformations



$\{X\}_{NP1-NP4}$	member 1	
	0	Gdof 1
	0	Gdof 2
	0.056160345	Gdof 3
	-0.05918947	Gdof 4

$\{e\}_e = [A]_e^T\{X\}_{NP1-NP4}$	
	member 1
	0
	-0.013655369

$\{F\}_e = [S]_e\{e\}_e$	
	member 1
	4.400060272
	-4.400060272



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Lesson 1 (Part 2) Fundamental Finite Element Method & Modeling

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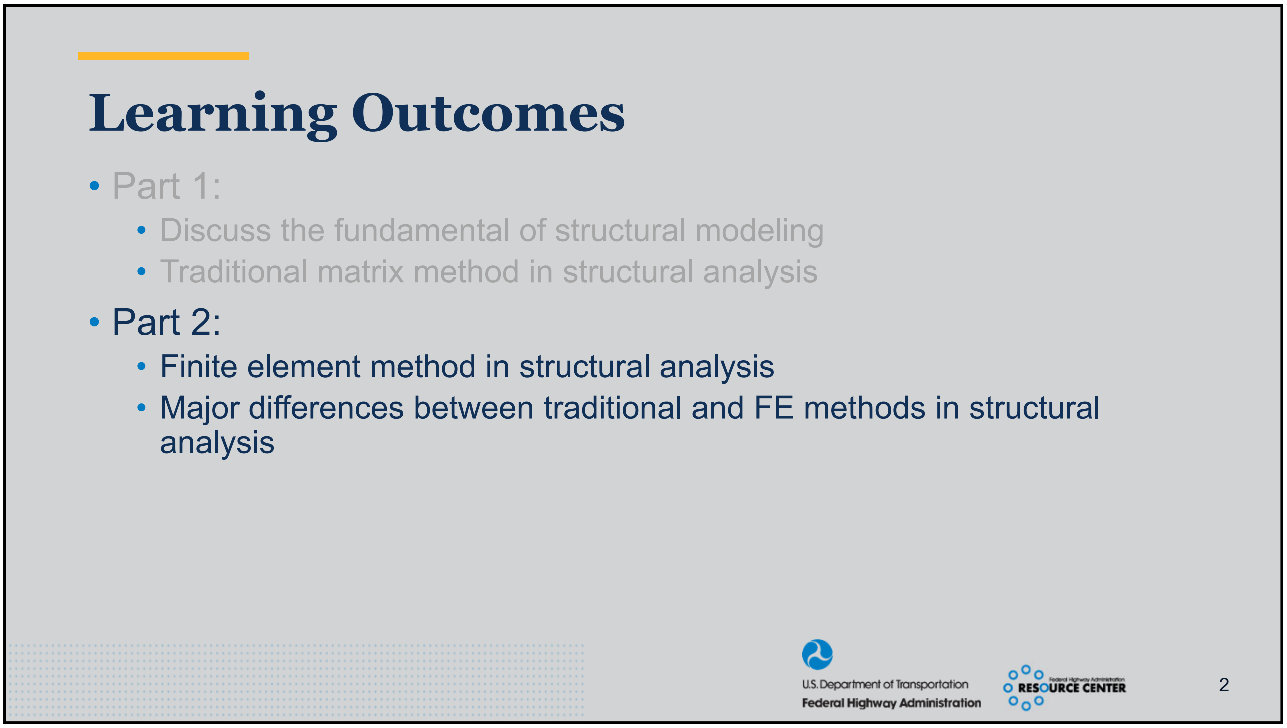

Learning Outcomes

- Part 1:
  - Discuss the fundamental of structural modeling
  - Traditional matrix method in structural analysis
- Part 2:
  - Finite element method in structural analysis
  - Major differences between traditional and FE methods in structural analysis

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Traditional matrix method in structural analysis

- Advantages:
  - Based on conventional mechanics of materials
  - Close-formed element stiffness matrix formulation
  - Member nodal forces & deformations are calculated directly
- Disadvantages:
  - Member stress & strain can not be calculated directly
  - The close-formed stiffness matrices for 2-D (plate & shell) and 3-D (solid) elements are very difficult to derived, based on conventional mechanics of materials.
  - Can not handle stress concentration problems and capture the 2-D or 3-D stress field.

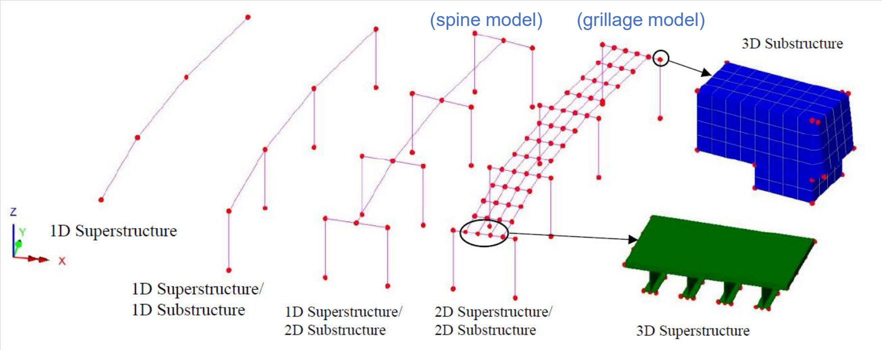
Finite element method in structural analysis

- Structural Modeling: same as traditional structural modeling
- Advantages:
  - Member stress & strain are calculated directly
  - The stiffness matrices for 2-D (plate & shell) and 3-D (solid) elements are directly formulated in open-form, based on principal of virtual work.
  - Effectively handle stress concentration problems and capture the 2-D or 3-D stress field.
- Disadvantages:
  - Much more finite elements are needed to capture the realistic structural response
  - Generation of input data file is time consuming.

Finite element method in structural analysis

- Element types
  - 1-D: Conventional truss, beam, spring, and point elements
  - 2-D: Plate and shell quadrilateral (or triangular) elements
  - 3-D: Solid hexahedra element
- Finite element stiffness matrix formulation
  - Step 1: Member deformation-nodal displacement relationship (by shape function)
  - Step 2: Strain-nodal displacement compatibility
  - Step 3: Material stress-strain relationship
  - Step 4: Principal of virtual work
  - Step 5: Gaussian points for numerical integration

Finite element method in structural analysis



(Traditional method) (FE method)

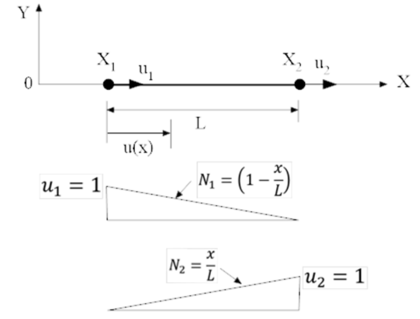
Concept of Finite Element stiffness formulation (FHWA Manual 2.3.2)

- Member deformation-nodal displacement relationship
  - by shape function (N_1, N_2)

$$u(x) = N_1 u_1 + N_2 u_2$$

$$= \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$

$$= \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{u\} \qquad (1)$$



$u(x)$: axial deformation along the element

Concept of Finite Element stiffness formulation

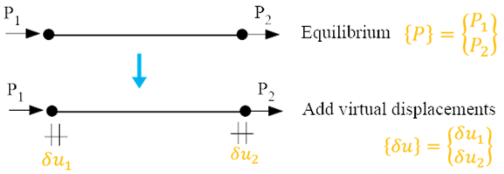
- Strain-nodal displacement compatibility

$$\varepsilon(x) = \frac{d}{dx} u(x) = \left(\frac{d}{dx}[N]\right)\{u\} = [B]\{u\} \qquad (2)$$

- Material stress-strain relationship

$$\sigma(x) = E\varepsilon(x) = E[B]\{u\} \qquad (3)$$

Concept of Finite Element stiffness formulation



- Principle of virtual work
  - External work:
    - $\delta W = \{\delta u\}^T\{P\}$ (4)
  - Internal virtual strain energy:
    - $\delta U = \int_v \delta\varepsilon(x)\sigma(x)dv$; where $\delta\varepsilon(x) = [B]\{\delta u\}$ from (2)
    - $= \int_v (\{\delta u\}^T[B]^T)(E[B]\{u\})dv$
    - $= \{\delta u\}^T[\int_v [B]^T E[B]\,dv]\,\{u\}$ (5)
  - $\delta W = \delta U$:
    - $\{\cancel{\delta u}\}^T\{P\} = \{\cancel{\delta u}\}^T[\int_v [B]^T E[B]\,dv]\,\{u\}$
    - $\therefore \{P\} = [k]\{u\}; where\ [k] = \int_v [B]^T E[B]\,dv$

FE Stiffness matrix formulation for truss element

- Formulate stiffness matrix $[k]$:
  - $[k] = \int_v [B]^T E[B]\,dv = \int_{x=0}^{x=L}[B]^T E[B]\,(Adx)$
  - $[B] = \frac{d}{dx}[N] = \frac{d}{dx}\left[1-\frac{x}{L} \quad \frac{x}{L}\right] = \left[\frac{-1}{L} \quad \frac{1}{L}\right]$
  - $[B]^T E[B] = E[B]^T\,[B] = E\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\end{bmatrix}\begin{bmatrix}\frac{-1}{L} & \frac{1}{L}\end{bmatrix} = E\begin{bmatrix}\frac{1}{L^2} & \frac{-1}{L^2}\\ \frac{-1}{L^2} & \frac{1}{L^2}\end{bmatrix}$
  - $\therefore\ [k] = \int_{x=0}^{x=L}[B]^T E[B]\,(Adx) = \begin{bmatrix}\frac{EA}{L^2}\int_0^L dx & \frac{-EA}{L^2}\int_0^L dx\\ \frac{-EA}{L^2}\int_0^L dx & \frac{EA}{L^2}\int_0^L dx\end{bmatrix} = \begin{bmatrix}\frac{EA}{L} & \frac{-EA}{L}\\ \frac{-EA}{L} & \frac{EA}{L}\end{bmatrix}$

Knowledge Review -- Question 1 (L1B S11 Q1)

The Finite Element stiffness matrix formulation is based on

A. Assumed member shape function, $[N]$

B. Strain-nodal displacement compatibility, $[B]$

C. Material stress-strain relationship, E

D. Principal of virtual work, $\delta W = \delta U$

E. All of the above

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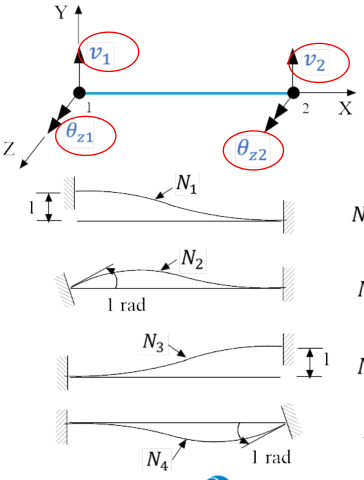
FE Stiffness matrix formulation for beam element (in Cartesian coordinate)

- Member deformation-nodal displacement relationship
  - by shape function (N_1, N_2, N_3, N_4) (FHWA Manual 2.3.2)

$$v(x) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

$$= [N]_{1x4}\{q\}_{4x1}$$



$$N_1 = \frac{1}{L^3}[2x^3 - 3x^2L + L^3]$$

$$N_2 = \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]$$

$$N_3 = \frac{1}{L^3}[-2x^3 + 3x^2L]$$

$$N_4 = \frac{1}{L^3}[x^3L - x^2L^2]$$

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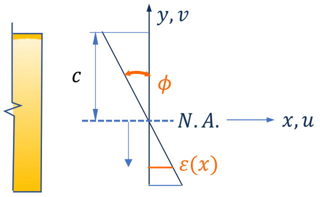
FE Stiffness matrix formulation for beam element

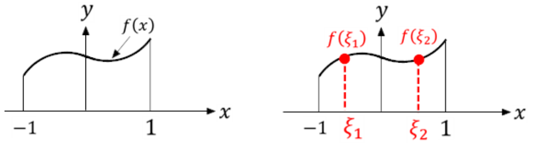
- Strain-nodal displacement compatibility
  - $\theta = \frac{dv}{dx}$; $\phi = \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$
  - $\varepsilon(x) = -y\phi = -y\frac{d^2v}{dx^2}$
  - $= -y\,\frac{d^2}{dx^2}([N]\{q\}) = \boxed{[B]}\{q\}$ (2)
- Material stress-strain relationship
  - $\sigma(x) = E\varepsilon(x) = E[B]\{q\}$ (3)



FE Stiffness matrix formulation for beam element

- Formulate stiffness matrix $[k]$ by principle of virtual work
  - $[k] = \int_v [B]^T E[B]\,dv = \int_0^L \int_A [B]^T E[B]dAdx$
  - $[B] = -y\,\frac{d^2}{dx^2}[N] = \frac{-y}{L^3}[12x - 6L \quad 6xL - 4L^2 \quad -12x + 6L \quad 6xl - 2L^2]$
  - $\therefore\ [k] = \int_0^L \int_A [B]^T E[B]dAdx = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ & \frac{4EI}{L} & \frac{12EI}{L^3} & \frac{2EI}{L} \\ & symm. & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ & & & \frac{4EI}{L} \end{bmatrix}; where\ I = \int y^2 dA$
- Disadvantage:
  - mathematic integration is tedious
  - not practical for computer application

Gaussian points for numerical integration between (-1, 1) (FHWA Manual 2.3.2)



$A = \int_{-1}^{1} f(x)dx$

$A = \sum_{i=1}^{n} W_i f(\xi_i) = W_1 f(\xi_1) + W_2 f(\xi_2)$

$n = total\ \#\ of\ Gauss\ points$

$\xi_i = ith\ Gauss\ point$

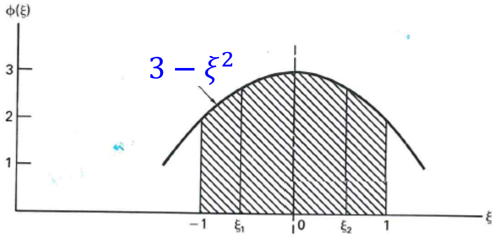
$W_i = weight\ of\ ith\ Gauss\ point$

Natural (or normal) coordinate: (-1, 1)

- Total # of Gaussian points needed, n:
  - $2n - 1 \geq m$
  - $m = degree\ of\ polynomial$

n	$\pm\xi_i$	W_i
1	0.0	2.0
2	0.577350	1.0
3	0.774597 0.0	0.55555 0.88888
4	0.8612136 0.3399810	0.347855 0.652145

Example: Find shading area based on natural coordinates

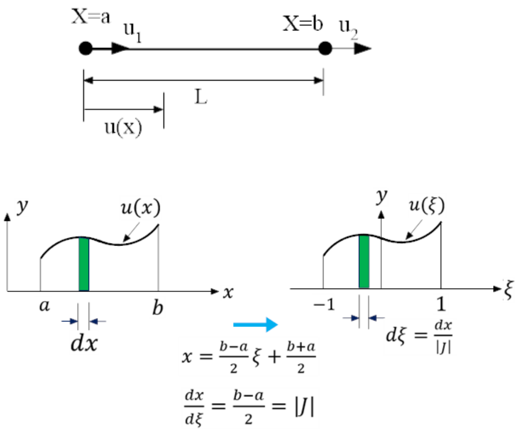


- By mathematic integration:
- A = $\int_{-1}^{1}(3-\xi^2)d\xi = 3\xi|_{-1}^{1} - \frac{\xi^3}{3}|_{-1}^{1}$ $= 5.333$

- By numerical integration:
  - Polynomial degree, $m = 2$
  - Total # of Gaussian points needed, n:
  - $2n - 1 \geq m = 2 \therefore n = 2$
  - $\therefore\ \xi_1 = -0.5773; \xi_2 = 0.5773$
  - A = $\sum_{i=1}^{n=2} W_i(3-\xi_i^2) = 1(3 - (-0.5773)^2) + 1(3 - (0.5773)^2) = 5.333$

Gaussian points for numerical integration (FHWA Manual 2.3.2)

- $Transfer\ x\ to\ \xi$
- $\therefore dx = |J| d\xi$
- Note: The term $|J|$ is called Jacobian Transformation
- (i.e. $|J|$ transfers Natural Coordinate to Cartesian Coordinate)



$d\xi = \frac{dx}{|J|}$

$x = \frac{b-a}{2}\xi + \frac{b+a}{2}$

$\frac{dx}{d\xi} = \frac{b-a}{2} = |J|$

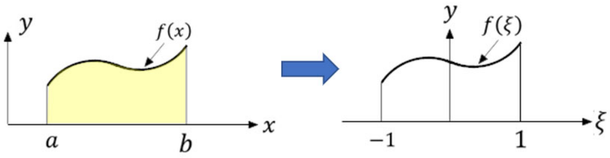
Gaussian points for numerical integration

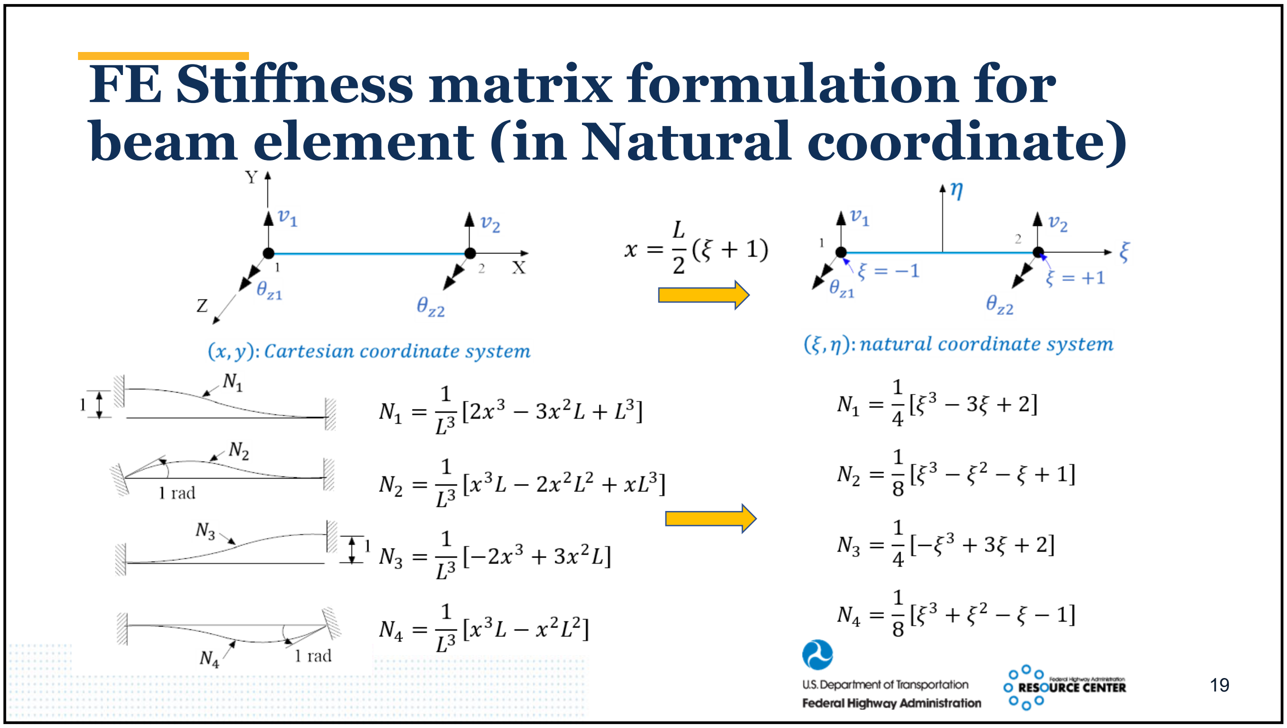
- $A = \int_a^b f(x)dx$

$$= \int_{-1}^{1} f(\xi)|J|d\xi$$

$$= |J| \int_{-1}^{1} f(\xi)d\xi$$

$$= |J| \sum_{i=1}^{n} W_i f(\xi_i)$$





FE Stiffness matrix formulation for beam element (in Natural coordinate)

$$x = \frac{L}{2}(\xi + 1)$$

(x, y): *Cartesian coordinate system*

(ξ, η): *natural coordinate system*

$$N_1 = \frac{1}{L^3}[2x^3 - 3x^2L + L^3] \qquad N_1 = \frac{1}{4}[\xi^3 - 3\xi + 2]$$

$$N_2 = \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3] \qquad N_2 = \frac{1}{8}[\xi^3 - \xi^2 - \xi + 1]$$

$$N_3 = \frac{1}{L^3}[-2x^3 + 3x^2L] \qquad N_3 = \frac{1}{4}[-\xi^3 + 3\xi + 2]$$

$$N_4 = \frac{1}{L^3}[x^3L - x^2L^2] \qquad N_4 = \frac{1}{8}[\xi^3 + \xi^2 - \xi - 1]$$

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FE Stiffness matrix formulation for beam element (in Natural coordinate)

- Formulate stiffness matrix $[k]$ by Gaussian quadrature
- $[K] = \int_V [B]^T E[B] dv$
- $= \int_V [B']^T E[B'] \, dA|J|d\xi$
- $= |J| \sum_{i=1}^{n} W_i \, [B']^T E[B']$
- where $[B]$ *is converted to* $[B']$ in the natural coordinate system

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Stress & strain distribution within the FE?

- Finite element relationships:
  - Equilibrium: $\{P\} = [K]\{q\}$
  - Compatibility: $\varepsilon(x) = [B]\{q\}$
  - Constitutive (stress-strain): $\sigma(x) = E\varepsilon(x)$
- Advantage: Stresses at Gaussian and nodal points can be directly obtained
  - $\sigma_i(x) = E\varepsilon_i(x) = E[B]_i\{q\}$; where $i = 1, n$

Element nodal force

Internal stress

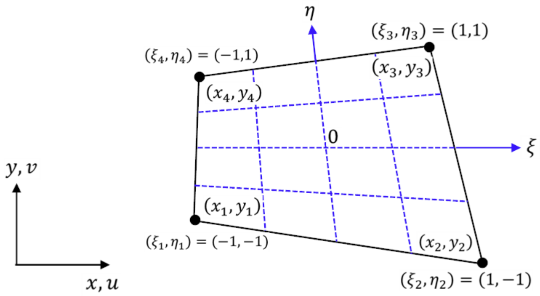
$$\begin{tikzcd}
\{P\} & \sigma(x) \\
\{q\} \arrow[u, "{[K]}"] \arrow[r, "{[B]}"'] & \varepsilon(x) \arrow[u, "E"']
\end{tikzcd}$$

Element nodal displacements

Internal strain

FE Stiffness matrix formulation for quadrilateral 2D plate element

- Member deformation-nodal displacement relationship
  - by shape function: (N_1, N_2, N_3, N_4)
  - $x = \sum_{i=1}^{4} N_i x_i; y = \sum_{i=1}^{4} N_i y_i;$ where $N_i = N_i(\xi)N_i(\eta)$ ⟸ 2D



(ξ, η): *natural coordinate system*

(Q4 element)

$(\xi_1, \xi_2, \xi_3, \xi_4) = (-1,1,1,-1)$
$(\eta_1, \eta_2, \eta_3, \eta_4) = (-1,-1,1,1)$

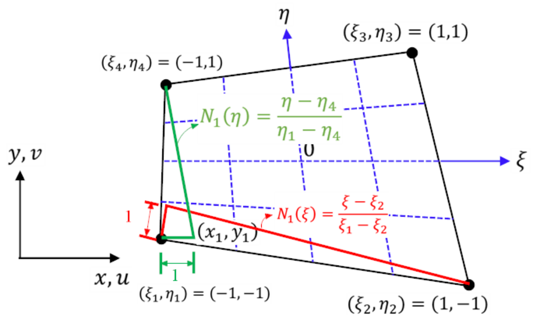
FE Stiffness matrix formulation for quadrilateral 2D plate element

- For example:

$$N_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1-1}$$

$$N_1(\eta) = \frac{\eta - \eta_4}{\eta_1 - \eta_4} = \frac{\eta - 1}{-1-1}$$

$$N_1 = N_1(\xi)N_1(\eta) = \frac{\xi - 1}{-2} \times \frac{\eta - 1}{-2}$$



- $u = \sum_{i=1}^{4} N_i u_i; v = \sum_{i=1}^{4} N_i v_i$
  (isoparametric element)

FE Stiffness matrix formulation for quadrilateral 2D plate element

- Strain-nodal displacement compatibility

- $$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_{3x1} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x}\sum N_i u_i \\ \frac{\partial}{\partial y}\sum N_i v_i \\ \frac{\partial}{\partial y}\sum N_i u_i + \frac{\partial}{\partial x}\sum N_i v_i \end{Bmatrix} = \boxed{[B]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [B]\{q\}$$

- $[B(x,y)]$ needs to be transferred to $[B(\xi,\eta)]$

FE Stiffness matrix formulation for quadrilateral 2D plate element

- Material stress-strain relationship
  - $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [E] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [E][B]\{q\}$

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FE Stiffness matrix formulation for quadrilateral 2D plate element

- Formulate stiffness matrix $[k]$ by Gaussian quadrature
- $\therefore [k] = \int_v [B]^T[E][B]dv$

  $= \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} W_i W_j [B]_{ij}^T [E][B]_{ij} \left| J_{ij} \right|$ (by Gaussian quadrature)

  where $[B]_{ij}$ is the $[B]$ matrix corresponding to Gaussian point (ξ_i, η_j)

  t_{ij}= plate thickness at Gaussian point (ξ_i, η_j)
- Note: $dv = tdxdy = t|J|d\xi d\eta$; $[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$ (Jacobian matrix)
- $[J]$ is to convert (x, y) to (ξ, η)

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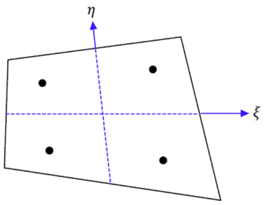
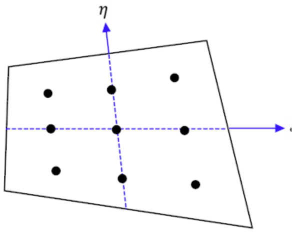
FE Stiffness matrix formulation for quadrilateral 2D plate element

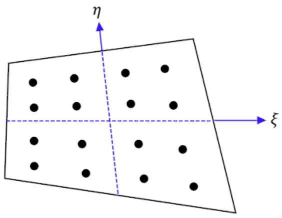
- Formulate stiffness matrix $[k]$ by Gaussian quadrature
- Total number of Gaussian points



Q4 element ($n = 2$ *each way*)



Q9 element ($n = 3$)



Q16 element ($n = 4$)

FE Stiffness matrix formulation for quadrilateral 2D plate element

- Find stress at each Gaussian point (ξ_i, η_j)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{ij} = [E][B]_{ij}\{q\} = [E][B]_{ij} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

- Find principal stresses at each Gaussian point (ξ_i, η_j)
  - $\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$
  - $\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$; $\alpha = \sigma_{max}\ direction$

Knowledge Review -- Question 2 (L1B S29 Q2)

What is the advantage of using Natural Coordinates (-1, 1) for the Finite Element stiffness matrix formulation?

A. Avoid tedious mathematic integration in Cartesian Coordinates

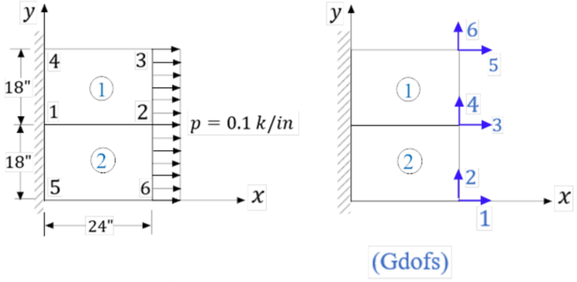
B. Perform numerical integration by using Gauss points

C. Obtained stresses directly at Gauss points.

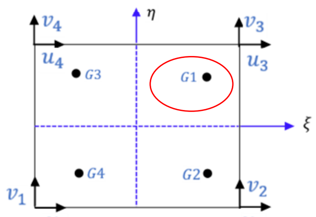
D. All of the above

Step-by-step example (see Appendix A)

- Find the $[K]_e$ of each element
- Formulate structural $[K]$
- Find principal stress at Gauss point G1 in element 1



$E = 30 \times 10^6\ psi$
$\mu = 0.25$
$t\ (thickness) = 0.1\ in$

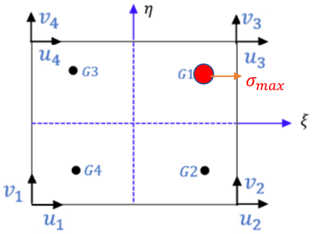


(Q4 Element ECS)

Step-by-step example (see Appendix A)

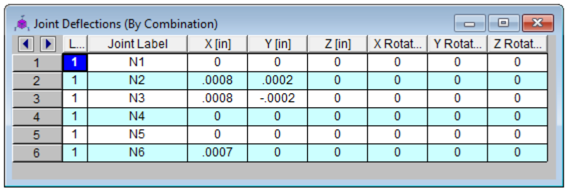
Based on hand calculation, the maximum principal stress at Gauss point G1 in element 1 is:

- $\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$
  $= 1006.92\ psi$
- $\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.069$
  $\therefore\ \alpha = -1.08°$



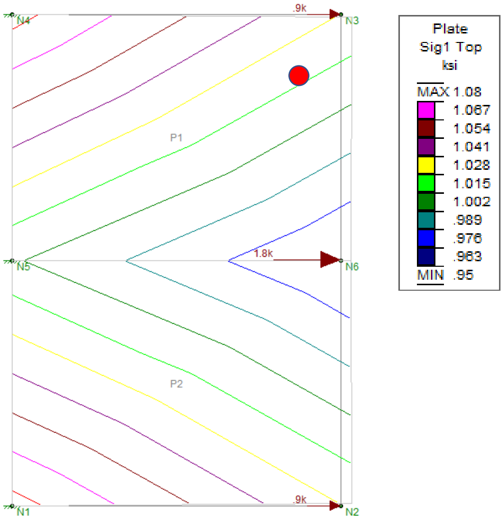
See Appendix A

Result from a commercial program:

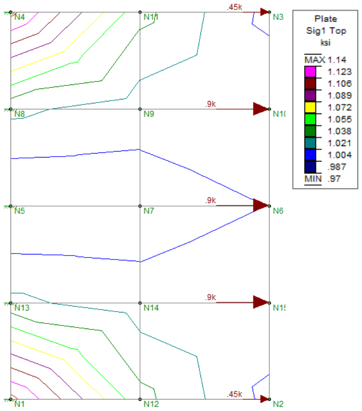


Joint Deflections (By Combination)

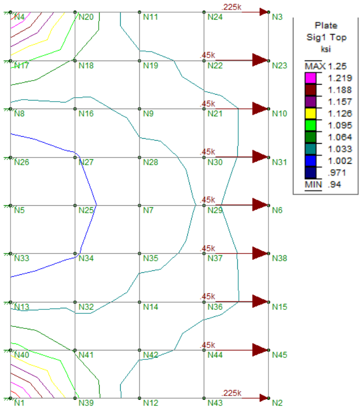
	L...	Joint Label	X [in]	Y [in]	Z [in]	X Rotat...	Y Rotat...	Z Rotat...
1	1	N1	0	0	0	0	0	0
2	1	N2	.0008	.0002	0	0	0	0
3	1	N3	.0008	-.0002	0	0	0	0
4	1	N4	0	0	0	0	0	0
5	1	N5	0	0	0	0	0	0
6	1	N6	.0007	0	0	0	0	0



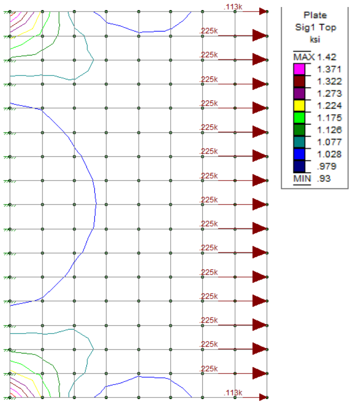
See Appendix A



8 elements



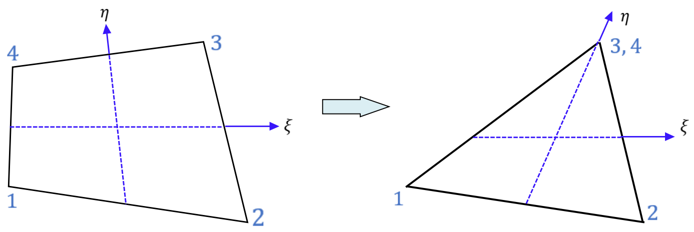
32 elements



128 elements

FE Stiffness matrix formulation for 2D triangular element (FHWA Manual 3.5.2.1)

- A special case of quadrilateral Q4 element
  - with $x_3 = x_4;\ y_3 = y_4$

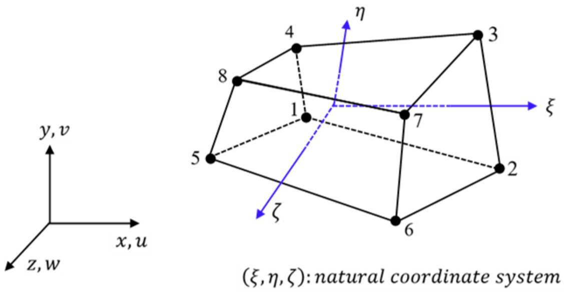


FE Stiffness matrix formulation for hexahedron 3D element (FHWA Manual 3.6.1.4)

- Member deformation-nodal displacement relationship
  - by interpolation (shape) function: $(N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8)$
  - $x = \sum_{i=1}^{8} N_i x_i;$
  - $y = \sum_{i=1}^{8} N_i y_i;$
  - $z = \sum_{i=1}^{8} N_i z_i;$

    where

    $N_i = N_i(\xi) N_i(\eta) N_i(\zeta)$ ⟸ 3D
  - $u = \sum_{i=1}^{8} N_i u_i;$
  - $v = \sum_{i=1}^{8} N_i v_i;$
  - $w = \sum_{i=1}^{8} N_i w_i;$



(ξ, η, ζ): natural coordinate system
(H8 element)

FE Stiffness matrix formulation for hexahedron 3D element

- Strain-nodal displacement compatibility
  - $\{\varepsilon\}_{6\times1} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\}^T$
  - $\{\varepsilon\}_{6x1} = [B]_{6x24}\{q\}_{24x1}$
- Material stress-strain relationship
  - $\{\sigma\}_{6x1} = [E]_{6x6}[B]_{6x24}\,\{q\}_{24x1}$
  - $\{\sigma\}_{6x1} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}^T$
- Formulate stiffness matrix $[k]$ by Gaussian quadrature
  - $\therefore [k] = \int_v [B]^T [E][B]\, dv$

    $= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} W_i\, W_j W_k [B]_{ijk}^T [E][B]_{ijk} |J_{ijk}|$

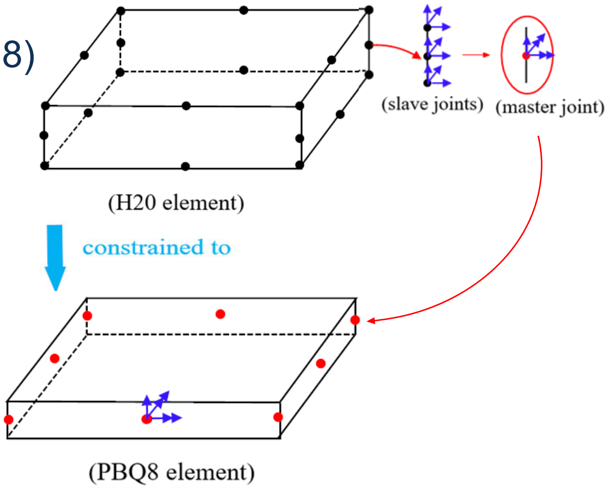
    (by Gaussian quadrature)

FE Stiffness matrix formulation for hexahedron 3D element

- Find stress at each Gaussian point (ξ_i, η_j, ζ_k)
  - $\{\sigma\}_{i,j,k,6x1} = [E]_{6x6}[B]_{i,j,k,6x24}\ \{q\}_{24x1}$
- Find principal stresses at each Gaussian point (ξ_i, η_j, ζ_k)

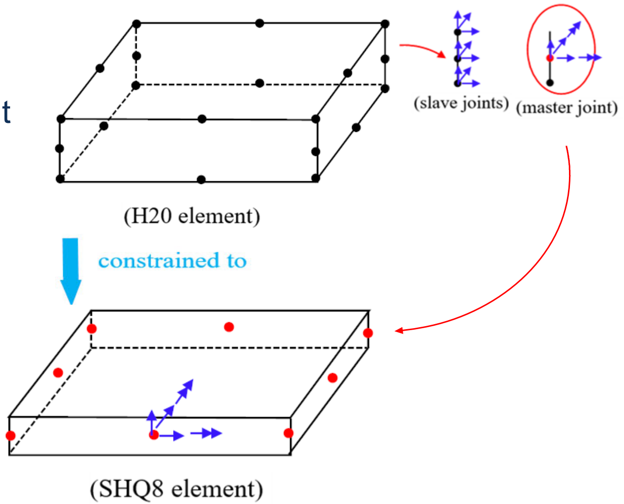
Other Finite Elements

- Plate bending element (PBQ 8)
  - Constrained from H20 element
  - Joint dofs:
    - 1 translational
    - 2 rotational



Other Finite Elements

- Shell element (SHQ 8):
  - Constrained from H20 element
  - Joint dofs:
    - 3 translational
    - 2 rotational



Modeling Loads

Joint load:

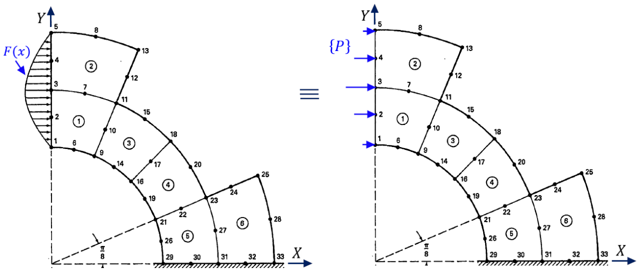
- Load directly applied at a joint

Element load:

- Transferred to the joints (i.e. Equivalent nodal loads)
- Through principal of virtual work (see Appendix C)
- $\{P\} = \int [N]^T F(x,y,z) dv$
- For example:

$\{P\} = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} W_i W_j [N]_{ij}^T F(x)_{ij} \left| J_{ij} \right|$

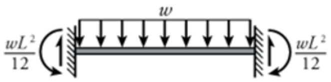
(by Gaussian quadrature)



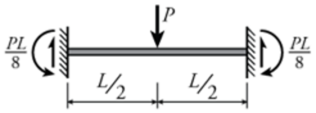
Example: Modeling Loads

Recall from Part 1:

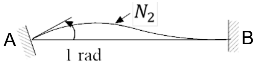
a) Distributed load



b) Concentrated load



Based on FEM:



$$N_2 = \frac{1}{L^3}[x^3L - 2x^2L^2 + xL^3]$$

a) Distributed load

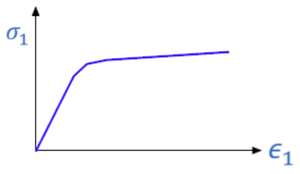
$$M_A = \int_0^l wN_2dx = \int_0^l w\left(\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right)dx = \frac{wL^2}{12}$$

b) Concentrated load

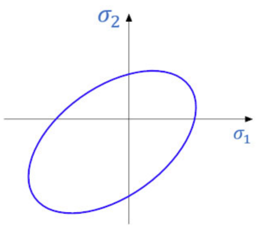
$$M_A = \int_0^l PN_2dx = P\left(\frac{x^3}{L^2} - \frac{2x^2}{L} + x\right)\Bigg|_{x\,=\,L/2} = \frac{PL}{8}$$

Inelastic Finite Element Analysis

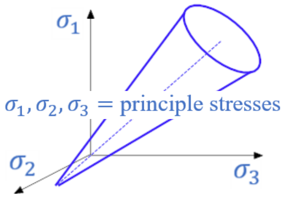
- Adjust $[E]$ per plastic flow rule for the yield curve or surface
- 1D element: $[k] = \int_{x=0}^{x=L}[B]^T\boxed{E}[B]\,(Adx)$
- 2D element: $[k] = \sum_{i=1}^{n}\sum_{j=1}^{n} W_iW_j[B]_{ij}^T\boxed{[E]}[B]_{ij}\left|J_{ij}\right|$
- 3D element: $[k] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} W_i\,W_jW_k[B]_{ijk}^T\boxed{[E]}[B]_{ijk}\left|J_{ijk}\right|$



1D: yield curve



2D: yield surface



3D: yield surface

Learning Outcome Review

- Finite element method in structural analysis
- Major differences between traditional and FE methods in structural analysis

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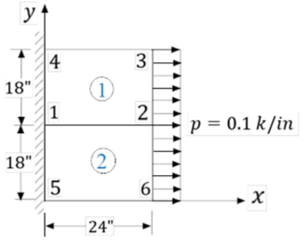
Next:
Lesson 2: General Girder Bridge Modeling

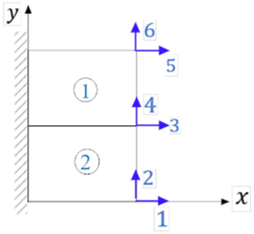
Appendix A

- Step-by-step example for the plate elements
  - Stiffness matrix formulation
  - Principal Stress analysis

Example

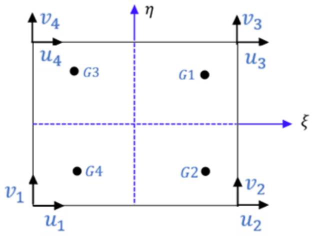
- Find the $[K]_e$ of each element
- Formulate structural $[K]$
- Find principal stresses of element 1





(Gdofs)

$E = 30 \times 10^6\ psi$
$\mu = 0.25$
$t\ (thickness) = 0.1\ in$



(Q4 Element ECS)

Example

- Calculate interpolation function

$$(\xi_1,\xi_2,\xi_3,\xi_4) = (-1,1,1,-1); (\eta_1,\eta_2,\eta_3,\eta_4) = (-1,-1,1,1)$$

$$N_i = N_i(\xi)N_i(\eta); \ N_1(\xi) = \frac{\xi-\xi_2}{\xi_1-\xi_2}; \ N_1(\eta) = \frac{\eta-\eta_4}{\eta_1-\eta_4}$$

$$N_1 = N_1(\xi)N_1(\eta) = \frac{\xi-1}{-1-1} \times \frac{\eta-1}{-1-1} = \frac{(1-\xi)(1-\eta)}{4}$$

$$N_2 = N_2(\xi)N_2(\eta) = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = N_3(\xi)N_3(\eta) = \frac{(1+\xi)(1+\eta)}{4}$$

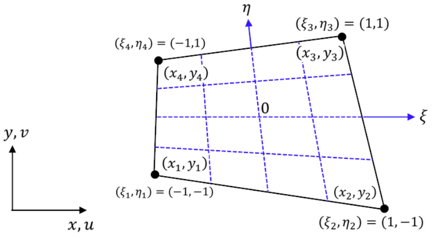
$$N_4 = N_4(\xi)N_4(\eta) = \frac{(1-\xi)(1-\eta)}{4}$$

$$\therefore x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

$$\therefore u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$



(ξ,η): *natural coordinate system*

(Q4 element)

Example

- Calculate $[B]$ matrix in Cartesian coordinate system

- $$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_{3x1} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x}\sum N_i u_i \\ \frac{\partial}{\partial y}\sum N_i v_i \\ \frac{\partial}{\partial y}\sum N_i u_i + \frac{\partial}{\partial x}\sum N_i v_i \end{Bmatrix} = [B]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{(Eq. A)}$$

- Since N_i is the function of (ξ,η), we need transfer $[B]$ to $[B]_{\xi,\eta}$ by Jacobian transformation

- $$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\sigma N_i}{\sigma \eta} \end{Bmatrix} = [J]\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \ \therefore \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1}\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}; [J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Example

- $[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$ matrix:
- $\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}(N_1x_1 + \ N_2x_2 + N_3x_3 + \ N_4x_4)$

  $= \frac{-(1-\eta)}{4}x_1 + \frac{(1-\eta)}{4}x_2 + \frac{(1+\eta)}{4}x_3 - \frac{(1+\eta)}{4}x_4$
- $\frac{\partial x}{\partial \eta} = \frac{-(1-\xi)}{4}x_1 - \frac{(1+\xi)}{4}x_2 + \frac{(1+\xi)}{4}x_3 + \frac{(1-\xi)}{4}x_4$
- $\frac{\partial y}{\partial \xi} = \frac{-(1-\eta)}{4}y_1 + \frac{(1-\eta)}{4}y_2 + \frac{(1+\eta)}{4}y_3 - \frac{(1+\eta)}{4}y_4$
- $\frac{\partial y}{\partial \eta} = \frac{-(1-\xi)}{4}y_1 - \frac{(1+\xi)}{4}y_2 + \frac{(1+\xi)}{4}y_3 + \frac{(1-\xi)}{4}y_4$

(Eq. B)

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Example 2

- $\therefore \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi}\sum N_i u_i \\ \frac{\partial}{\partial \eta}\sum N_i u_i \end{bmatrix}$
- $= \frac{1}{4}[J]^{-1} \begin{bmatrix} -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta & 0 \\ -1+\xi & 0 & 1-\xi & 0 & 1+\xi & 0 & -1-\xi & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$ (Eq. C1)

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Example

- $$\therefore \begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \sum N_i v_i \\ \frac{\partial}{\partial \eta} \sum N_i v_i \end{bmatrix}$$
- $$= \frac{1}{4}[J]^{-1} \begin{bmatrix} 0 & -1+\eta & 0 & 1-\eta & 0 & 1+\eta & 0 & -1-\eta \\ 0 & -1+\xi & 0 & 1-\xi & 0 & 1+\xi & 0 & -1-\xi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad \text{(Eq. C2)}$$

Example

Formulate element 1 stiffness matrix

- $(x_1, x_2, x_3, x_4) = (0,24,24,0)$
- $(y_1, y_2, y_3, y_4) = (18,18,36,36)$
- At 1st Gaussian Point $(\xi, \eta) = (0.57735, 0.57735)$

$$\left.\begin{array}{l} \frac{\partial x}{\partial \xi} = \frac{-(1-\eta)}{4} x_1 + \frac{(1-\eta)}{4} x_2 + \frac{(1+\eta)}{4} x_3 - \frac{(1+\eta)}{4} x_4 = 12 \\ \frac{\partial x}{\partial \eta} = \frac{-(1-\xi)}{4} x_1 - \frac{(1+\xi)}{4} x_2 + \frac{(1+\xi)}{4} x_3 + \frac{(1-\xi)}{4} x_4 = 0 \\ \frac{\partial y}{\partial \xi} = \frac{-(1-\eta)}{4} y_1 + \frac{(1-\eta)}{4} y_2 + \frac{(1+\eta)}{4} y_3 - \frac{(1+\eta)}{4} y_4 = 0 \\ \frac{\partial y}{\partial \eta} = \frac{-(1-\xi)}{4} y_1 - \frac{(1+\xi)}{4} y_2 + \frac{(1+\xi)}{4} y_3 + \frac{(1-\xi)}{4} y_4 = 9 \end{array}\right\} \quad \text{(Eq. B)}$$

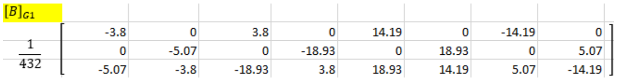
Example

- At 1st Gaussian Point G1$(\xi,\eta) = (0.57735, 0.57735)$

$$[J]_{G1} = \begin{bmatrix} 12 & 0 \\ 0 & 9 \end{bmatrix}; [J]_{G1}^{-1} = \frac{1}{108}\begin{bmatrix} 9 & 0 \\ 0 & 12 \end{bmatrix}$$

From (Eqs. A & C):

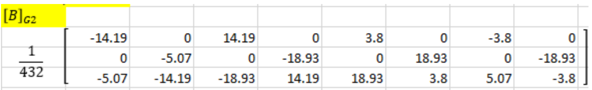
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_{G1} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix};$$



$$[B]_{G1} \quad \frac{1}{432}\begin{bmatrix} -3.8 & 0 & 3.8 & 0 & 14.19 & 0 & -14.19 & 0 \\ 0 & -5.07 & 0 & -18.93 & 0 & 18.93 & 0 & 5.07 \\ -5.07 & -3.8 & -18.93 & 3.8 & 18.93 & 14.19 & 5.07 & -14.19 \end{bmatrix}$$

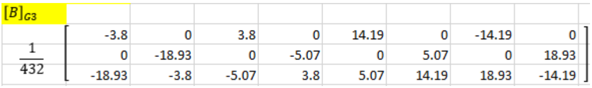
Example

- Similarly, at 2st Gaussian Point G2$(\xi,\eta) = (0.57735, -0.57735)$



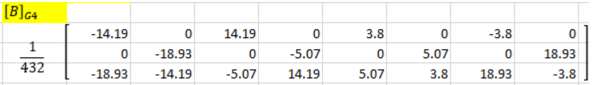
$$[B]_{G2} \quad \frac{1}{432}\begin{bmatrix} -14.19 & 0 & 14.19 & 0 & 3.8 & 0 & -3.8 & 0 \\ 0 & -5.07 & 0 & -18.93 & 0 & 18.93 & 0 & -18.93 \\ -5.07 & -14.19 & -18.93 & 14.19 & 18.93 & 3.8 & 5.07 & -3.8 \end{bmatrix}$$

- At 3rd Gaussian Point G3$(\xi,\eta) = (-0.57735, 0.57735)$



$$[B]_{G3} \quad \frac{1}{432}\begin{bmatrix} -3.8 & 0 & 3.8 & 0 & 14.19 & 0 & -14.19 & 0 \\ 0 & -18.93 & 0 & -5.07 & 0 & 5.07 & 0 & 18.93 \\ -18.93 & -3.8 & -5.07 & 3.8 & 5.07 & 14.19 & 18.93 & -14.19 \end{bmatrix}$$

- At 4th Gaussian Point G4$(\xi,\eta) = (-0.57735, -0.57735)$



$$[B]_{G4} \quad \frac{1}{432}\begin{bmatrix} -14.19 & 0 & 14.19 & 0 & 3.8 & 0 & -3.8 & 0 \\ 0 & -18.93 & 0 & -5.07 & 0 & 5.07 & 0 & 18.93 \\ -18.93 & -14.19 & -5.07 & 14.19 & 5.07 & 3.8 & 18.93 & -3.8 \end{bmatrix}$$

Example

- Formulate Element 1 Stiffness matrix, $[K]_{e1}$
- $[K]_{e1} = \sum_{i=1}^{2}\sum_{j=1}^{2} t_{ij} W_i W_j [B]_{ij}^{T}[E][B]_{ij}\left|J_{ij}\right|$

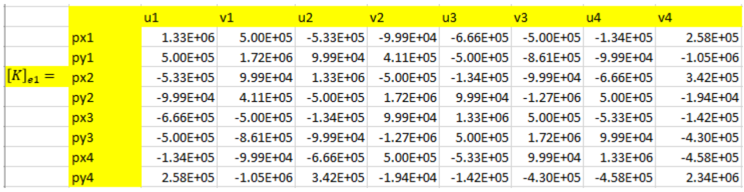
  where

  $W_i = W_j = 1;\ t_{ij} = t = 0.1\ in$

  $$[E] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{30\times10^6}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Example

- Formulate Element 1 Stiffness matrix, $[K]_{e1}$, in ECS
- $[K]_{e1} = \sum_{i=1}^{2}\sum_{j=1}^{2} t_{ij} W_i W_j [B]_{ij}^{T}[E][B]_{ij}\left|J_{ij}\right|$



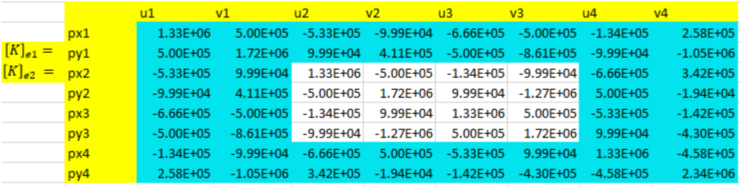
		u1	v1	u2	v2	u3	v3	u4	v4
	px1	1.33E+06	5.00E+05	-5.33E+05	-9.99E+04	-6.66E+05	-5.00E+05	-1.34E+05	2.58E+05
	py1	5.00E+05	1.72E+06	9.99E+04	4.11E+05	-5.00E+05	-8.61E+05	-9.99E+04	-1.05E+06
$[K]_{e1} =$	px2	-5.33E+05	9.99E+04	1.33E+06	-5.00E+05	-1.34E+05	-9.99E+04	-6.66E+05	3.42E+05
	py2	-9.99E+04	4.11E+05	-5.00E+05	1.72E+06	9.99E+04	-1.27E+06	5.00E+05	-1.94E+04
	px3	-6.66E+05	-5.00E+05	-1.34E+05	9.99E+04	1.33E+06	5.00E+05	-5.33E+05	-1.42E+05
	py3	-5.00E+05	-8.61E+05	-9.99E+04	-1.27E+06	5.00E+05	1.72E+06	9.99E+04	-4.30E+05
	px4	-1.34E+05	-9.99E+04	-6.66E+05	5.00E+05	-5.33E+05	9.99E+04	1.33E+06	-4.58E+05
	py4	2.58E+05	-1.05E+06	3.42E+05	-1.94E+04	-1.42E+05	-4.30E+05	-4.58E+05	2.34E+06

- Since element 2 has same shape & size as element 1,

  $[K]_{e2} = [K]_{e1}$

Example

- Formulate Element 1 Stiffness matrix, $[K]_{e1}$, in ECS
- $[K]_{e1} = \sum_{i=1}^{2} \sum_{j=1}^{2} t_{ij} W_i W_j [B]_{ij}^T [E][B]_{ij} \left| J_{ij} \right|$



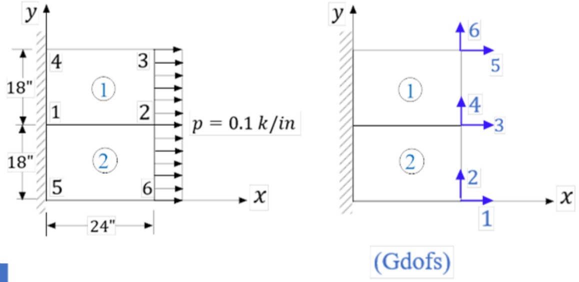
		u1	v1	u2	v2	u3	v3	u4	v4
	px1	1.33E+06	5.00E+05	-5.33E+05	-9.99E+04	-6.66E+05	-5.00E+05	-1.34E+05	2.58E+05
$[K]_{e1} =$	py1	5.00E+05	1.72E+06	9.99E+04	4.11E+05	-5.00E+05	-8.61E+05	-9.99E+04	-1.05E+06
$[K]_{e2} =$	px2	-5.33E+05	9.99E+04	1.33E+06	-5.00E+05	-1.34E+05	-9.99E+04	-6.66E+05	3.42E+05
	py2	-9.99E+04	4.11E+05	-5.00E+05	1.72E+06	9.99E+04	-1.27E+06	5.00E+05	-1.94E+04
	px3	-6.66E+05	-5.00E+05	-1.34E+05	9.99E+04	1.33E+06	5.00E+05	-5.33E+05	-1.42E+05
	py3	-5.00E+05	-8.61E+05	-9.99E+04	-1.27E+06	5.00E+05	1.72E+06	9.99E+04	-4.30E+05
	px4	-1.34E+05	-9.99E+04	-6.66E+05	5.00E+05	-5.33E+05	9.99E+04	1.33E+06	-4.58E+05
	py4	2.58E+05	-1.05E+06	3.42E+05	-1.94E+04	-1.42E+05	-4.30E+05	-4.58E+05	2.34E+06

- Since element 2 has same shape & size as element 1,

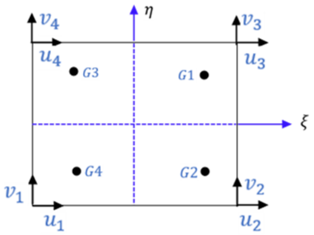
  $[K]_{e2} = [K]_{e1}$

Example

- Use direct element method, find global structural stiffness matrix:



Element	Pu2	Pv2	Pu3	Pv3
①	3	4	5	6
②	1	2	3	4



Direct Element Method - $[K]$

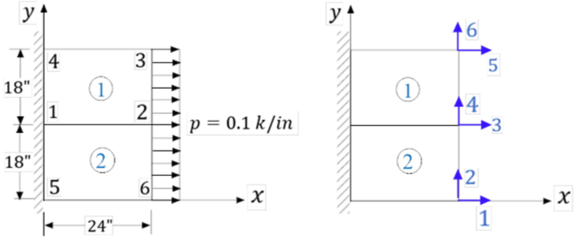
Map $[K]_e$ into the global stiffness, $[K]$, per the mapping table

		u1	v1	u2	v2	u3	v3	u4	v4	
	px1									
$[K]_{e1} =$	py1									
$[K]_{e2} =$	px2			1.334	-0.5	-0.133	-0.1			$\times 10^6$
	py2			-0.5	1.722	0.1	-1.272			
	px3			-0.133	0.1	1.334	0.5			
	py3			-0.1	-1.272	0.5	1.722			
	px4									
	py4									

Element	Pu2	Pv2	Pu3	Pv3
①	3	4	5	6
②	1	2	3	4

		1	2	3	4	5	6		
	1								
	2								
$[K] =$	3			2.668	0				$\times 10^6$
	4			0	3.444				
	5								
	6								

Example



(Gdofs)

		1	2	3	4	5	6		
	1	1.334	-0.5	-0.133	-0.1	0	0		
	2	-0.5	1.722	0.1	-1.272	0	0		
$[K] =$	3	-0.133	0.1	2.668	0	-0.133	-0.1		$\times 10^6$
	4	-0.1	-1.272	0	3.444	0.1	-1.272		
	5	0	0	-0.133	0.1	1.334	0.5		
	6	0	0	-0.1	-1.272	0.5	1.722		

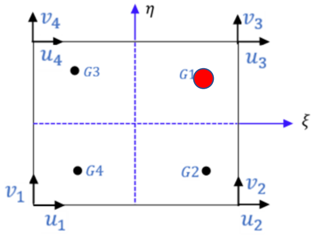
	0.974582	0.52525	0.033148	0.330585	-0.12761	0.283174	
	0.52525	1.261756	-0.01239	0.714528	-0.28317	0.609308	
$[K]^{-1} =$	0.033148	-0.01239	0.379046	-5.2E-18	0.033148	0.012387	$\times 10^{-6}$
	0.330585	0.714528	-6E-18	0.837363	-0.33059	0.714528	
	-0.12761	-0.28317	0.033148	-0.33059	0.974582	-0.52525	
	0.283174	0.609308	0.012387	0.714528	-0.52525	1.261756	

	900			821.93776			0.0008219
$\{P\} =$	0		$\{X\} =$	195.57139			0.0001956
	1800			741.94946	$\times 10^{-6}$	=	0.0007419
	0			0			0
	900			821.93776			0.0008219
	0			-195.5714			-0.000196

Example

Find principal stresses at Gauss point G1 in element 1:

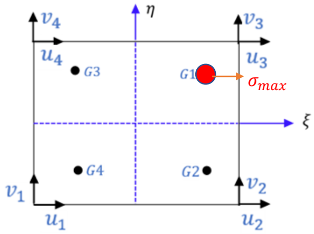
- G1$(\xi, \eta) = (0.57735, 0.57735)$
- $$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{G1} = [E][B]_{(0.5773,0.5773)} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0.000742 \\ v_2 = 0 \\ u_3 = 0.000822 \\ v_3 = -0.0002 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$
- $$= \begin{Bmatrix} 1005.71 \\ -5.834 \\ -35.08 \end{Bmatrix} \text{(psi)}$$



Example

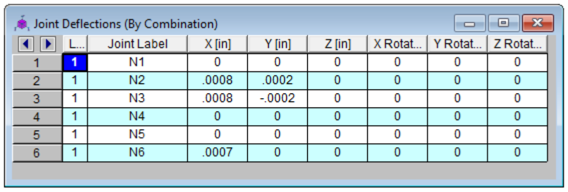
Find principal stresses at Gauss point G1 in element 1:

- $$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$
  $$= 1006.92\,psi$$
- $$\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.069$$
  $$\therefore\ \alpha = -1.08°$$

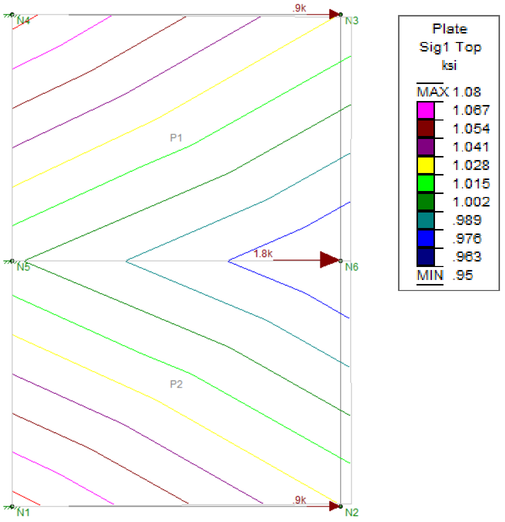


Example

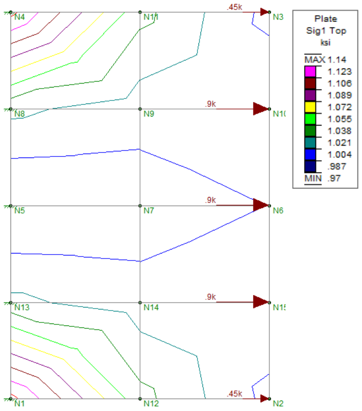
Result from a commercial program:

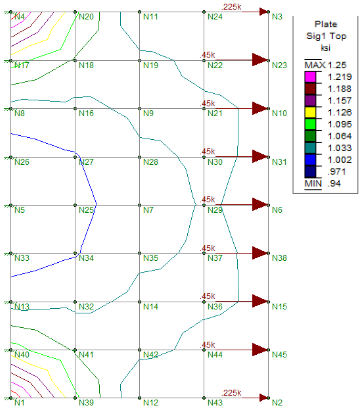


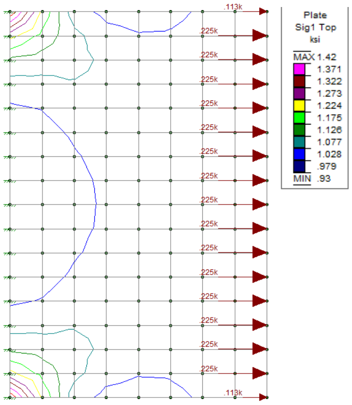
	L...	Joint Label	X [in]	Y [in]	Z [in]	X Rotat...	Y Rotat...	Z Rotat...
1	1	N1	0	0	0	0	0	0
2	1	N2	.0008	.0002	0	0	0	0
3	1	N3	.0008	-.0002	0	0	0	0
4	1	N4	0	0	0	0	0	0
5	1	N5	0	0	0	0	0	0
6	1	N6	.0007	0	0	0	0	0



Example







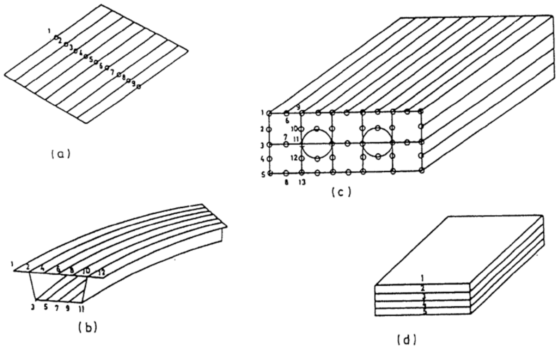
Appendix B: Other Discretized methods

- Finite Strip method
- Boundary Element method

Finite Strip method

No difference between FSM and FEM, except:

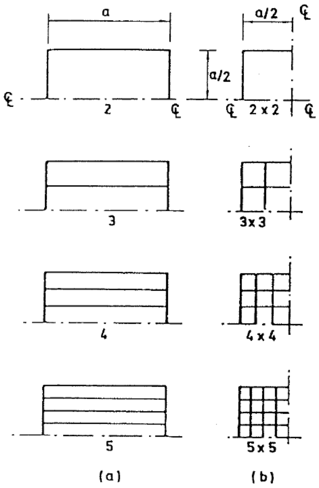
- FS element has a much longer strip in the element's long. direction.
- Stringent shape function in long. direction.
  - For example, from the solution of applicable classical differential equation



Finite Strip method

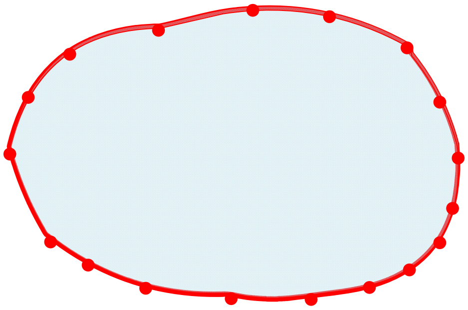
Compare with FEM:

- Small amount of mesh lines (elements) is needed.



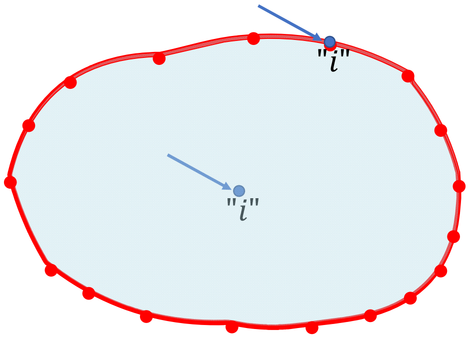
Boundary Element method

- Based on classical theory of elasticity, a high-order differential equation can be derived.
- Using integration by part to the differential equation, The boundary integral formulation can be derived.
- The boundary integration can be discretized into a serious boundary elements, so numerical integration can be performed.



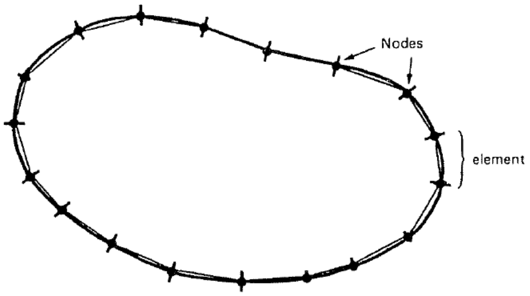
Boundary Element method

- When applying a load at any internal point "i" within or on the boundary of an elastic body
  - The force (stress) and displacement (strain) at any boundary points can be calculated
  - The stress and strain at any other internal points can be calculated



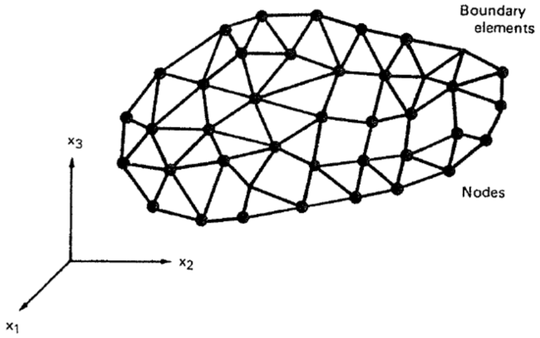
Boundary Element method

- 2-D elastic domain:
  - Boundary is 1-D (curve/straight line)
  - Discretize boundary curve into several 1-D line BEs



Boundary Element method

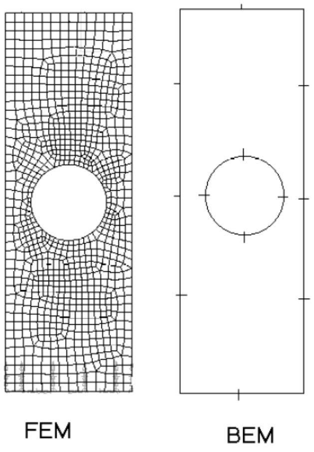
- 3-D elastic domain:
  - Boundary is 2-D (surface)
  - Discretize boundary surface into several 2-D plate BEs



Boundary Element method

Compare with FEM:

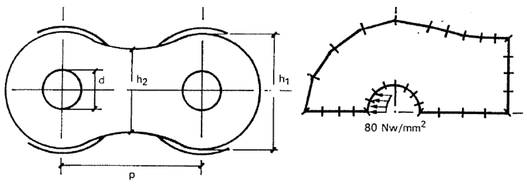
- Only require discretization of boundary rather than the elastic domain (element)
- Easier than FEM for analyzing stress concentration and fracture mechanics applications
  - Small amount of BEs is needed.



Boundary Element method

Compare with FEM:

- No discretization of interior is required



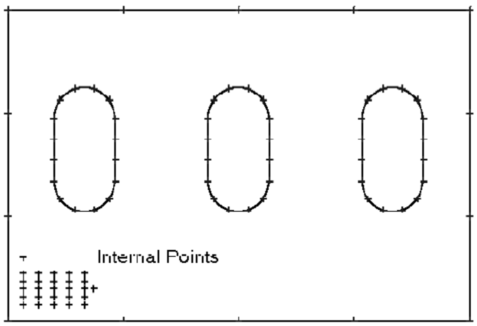


Plate with three holes

Appendix C: FE Modeling for Element distributed load

- Use Principle of Virtual Work
- Find equivalent nodal loads

FEM analysis for creep & shrinkage consideration

- Similar to FEM stiffness matrix formulation, the equivalent nodal forces, $\{P\}$, due to element distributed load can be obtained from the concept of Principle of virtual work, $\delta U = \delta W$ as follows:
- Let $\{q\}$ represent nodal displacement vector, $\mathrm{b}(x, y, z)$ represent distributed element force, and $\mathrm{u}(x, y, z)$ represent element deformation
  - $\delta U = \int \delta \varepsilon^T \sigma dv$ (1)
  - $\delta W = \int \delta u^T b dv$ (2)
  - Let (1)=(2)

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FEM analysis for creep & shrinkage consideration

- $\int \delta \varepsilon^T \sigma dv = \int \delta u^T b dv$
- $\rightarrow \int B^T \delta q^T \sigma dv = \int \delta q^T N^T b dv$; since $u = Nq$
- $\rightarrow \delta q^T (\int B^T E B dv)\, q = \delta q^T \int N^T b dv$; since $\sigma = E\varepsilon = E(Bq)$
- $\rightarrow \delta q^T [K] q = \delta q^T \int N^T b dv$
- ∴ equivalent nodal force $\mathrm{P} = [K]q = \int N^T b dv$

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Lesson 2 General Girder Bridge Modeling

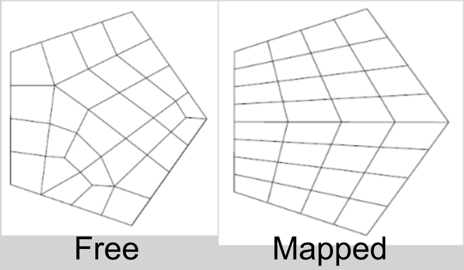
Learning Outcomes

- Explain the general steps and key parameters that affect building a good general bridge FEA model.
- Describe the basic procedures for 1D, 2D and 3D refined analysis and the recommended analysis tool.

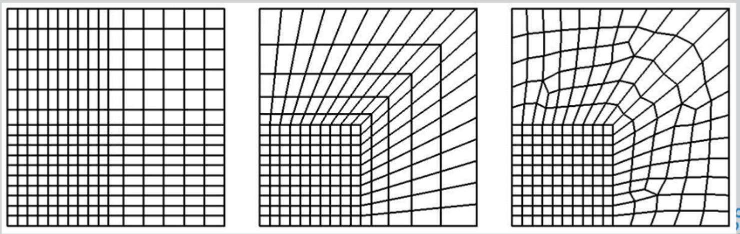
Basic Assumptions for Routine FEA

- Isotropic, linear elastic materials. Shear deformations are negligible.
- Torsional warping is small enough to be neglected.
- Sections do not change due to cracking or yielding during analysis
- Boundary conditions are considered either fully restrained or fully unrestrained.
- Restraints applied at bearing locations and joints between members.
- Loads remain constant in direction and magnitude. Neglect second-order effects and no geometric nonlinearities.
- Superposition of loads is valid since material or geometric nonlinearities are negligible.
- Bridge live loadings can be approximated with concentrated point and distributed loads

Meshing



- Element types
- Meshing of surfaces and volumes
- Regular, irregular, free, transition, and mapped meshes



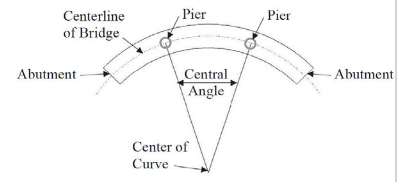
Secondary considerations to control meshing

- Locating nodes where loads are to be applied,
- Locating nodes where output is desired,
- Locating nodes at interfaces with adjacent elements, such as diaphragms, stiffeners, cross-frames, etc.
- Locating nodes where it is anticipated that a later iteration of the model will require a node, or
- Orienting the mesh in order to obtain stresses in a specific direction.

One Dimensional Analysis (1-D) Limitations

For I-girders:

- Girders are concentric
- Bearing lines are not skewed more than 10 degrees from radial
- The stiffness of the girders is similar
- The central angle is less than 0.06 radians, where the arc span L_{as}, is taken as:
  - 0.9 times the arc length for end spans
  - 0.8 times the arc length for interior spans

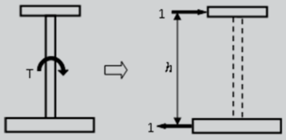


One Dimensional Analysis (1-D) Limitations

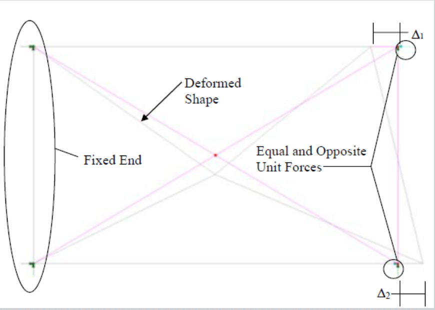
For box girders:

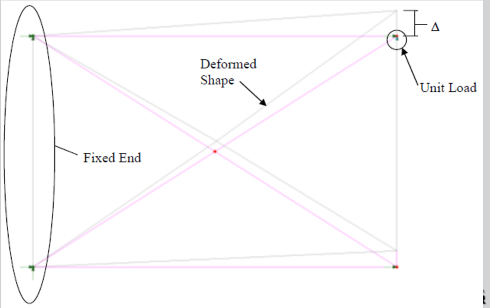
- Girders are concentric
- Bearing lines are not skewed
- The girder depth is less than the width of the box at mid-depth
- The central is < 0.3 radians, where the arc span Las, is:
  - 0.9 times the arc length for end spans
  - 0.8 times the arc length for interior spans
  - For concrete box, the central angle is < 12 deg (LRFD 4.6.1.2.3)

Property Calculation



- Section properties
- Warping torsion, equivalent St. Venant’s
- Diaphragm and cross-frame stiffness

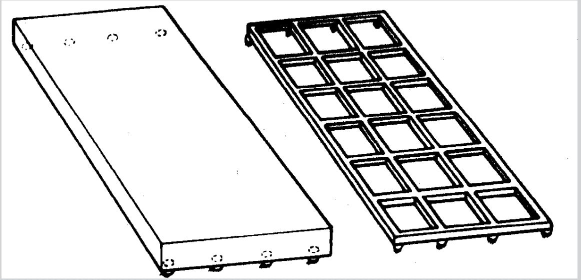




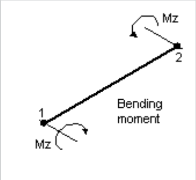
Two Dimensional Analysis

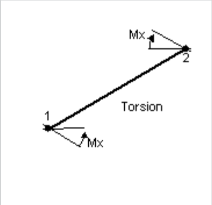
Grillage Analogy Method:

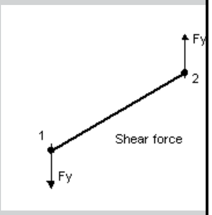
- Essentially a computer-aided method for analysis of bridge decks
- The deck is idealized as a series of ‘beam’ elements (or grillages), connected and restrained at their joints.



Solid Slab



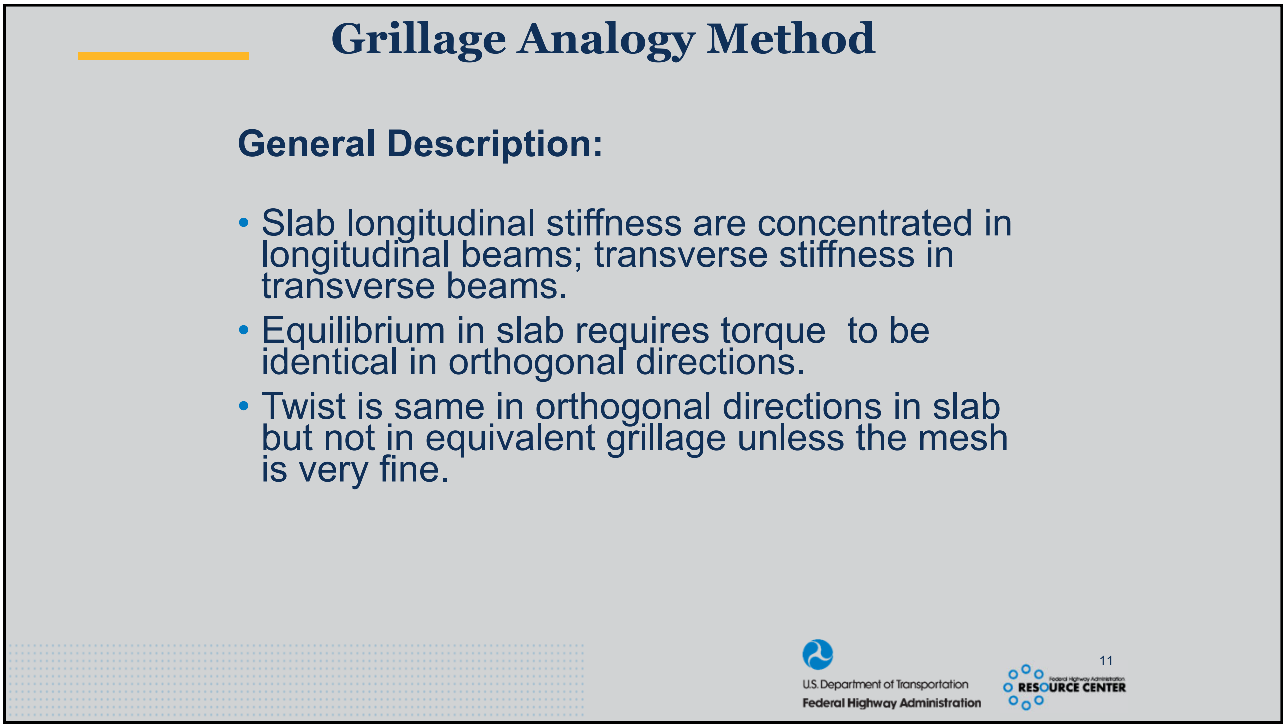




Grillage Analogy Method

General Description:

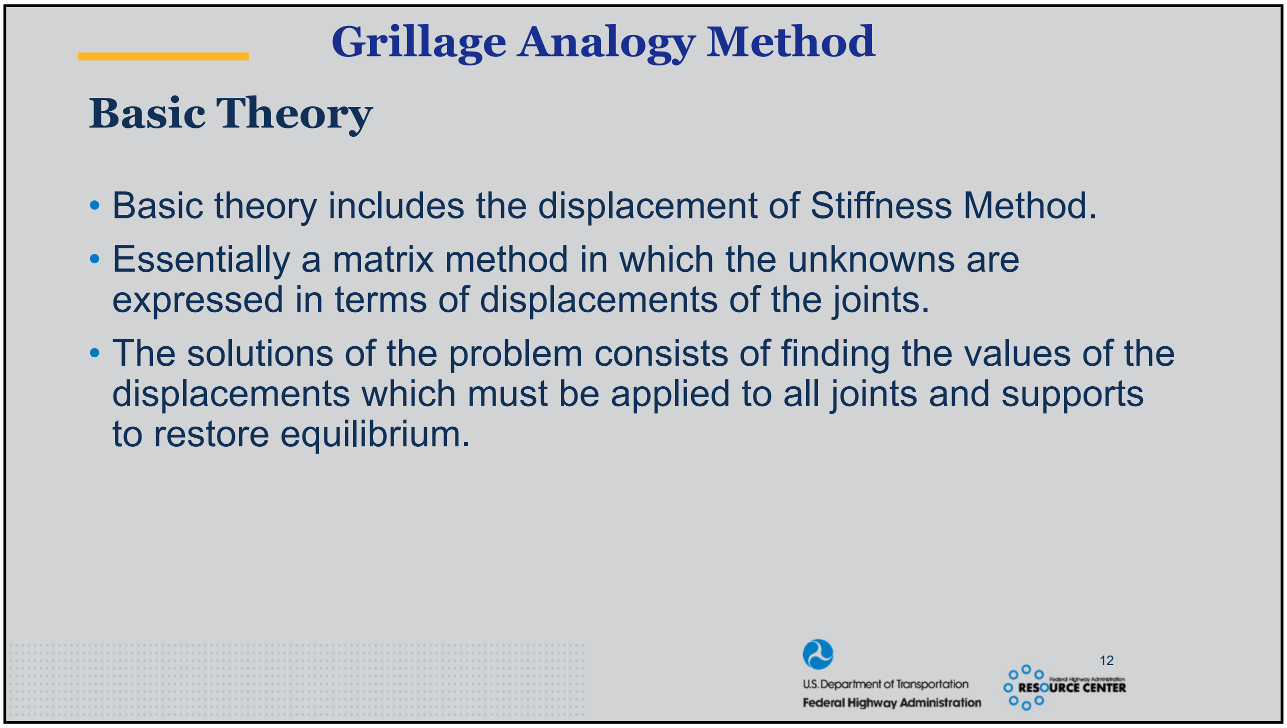
- Each element is given an equivalent bending and torsional inertia to represent the portion of the deck which it replaces.
- Bending and torsional stiffness in every region of slab are assumed to be concentrated in nearest equivalent grillage beam.
- Restraints, load and supports may be applied at the joints between the members, and members framing into a joint may be at any angle.



Grillage Analogy Method

General Description:

- Slab longitudinal stiffness are concentrated in longitudinal beams; transverse stiffness in transverse beams.
- Equilibrium in slab requires torque to be identical in orthogonal directions.
- Twist is same in orthogonal directions in slab but not in equivalent grillage unless the mesh is very fine.



Grillage Analogy Method

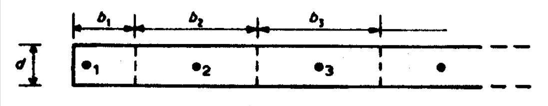
Basic Theory

- Basic theory includes the displacement of Stiffness Method.
- Essentially a matrix method in which the unknowns are expressed in terms of displacements of the joints.
- The solutions of the problem consists of finding the values of the displacements which must be applied to all joints and supports to restore equilibrium.

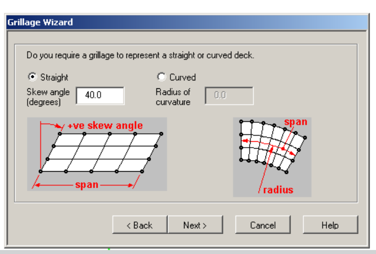
Grillage Analogy Method

Features (Solid Slab)

- Longitudinal member



  - must cross any discrete bearings
  - spacing closer than 0.25×span essential
  - spacing closer than 2d or 3d pointless
  - within 0.3d of edge of slab
- Transverse members
  - Ideally orthogonal but skews <20° OK
  - similar spacing to longitudinal



Do you require a grillage to represent a straight or curved deck.

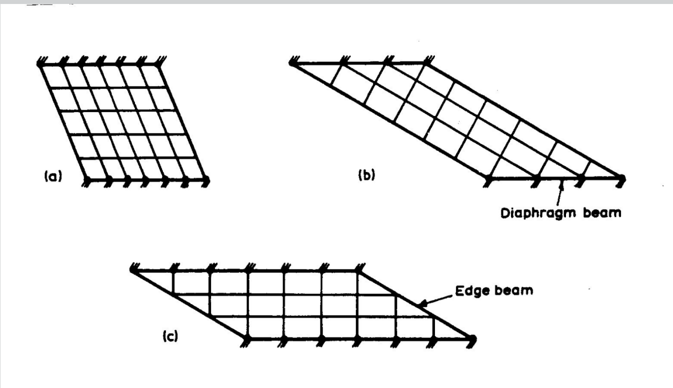
● Straight ○ Curved

Skew angle (degrees) 40.0

Radius of curvature 0.0

Grillage Analogy Method

Features (Skew Decks)

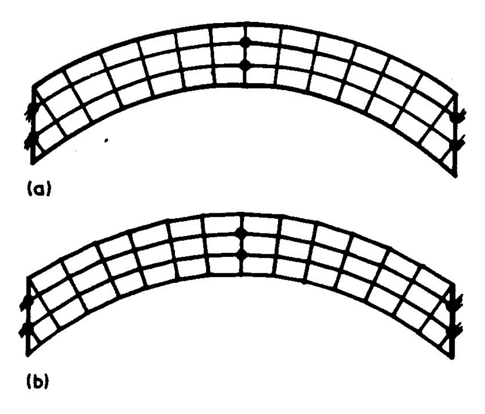


Grillages for skew decks

(a) skew mesh (b) mesh orthogonal to span (c) mesh orthogonal to support

Grillage Analogy Method

Features (curved decks)



Grillages for curved decks
(a) curved members (b) Straight segments

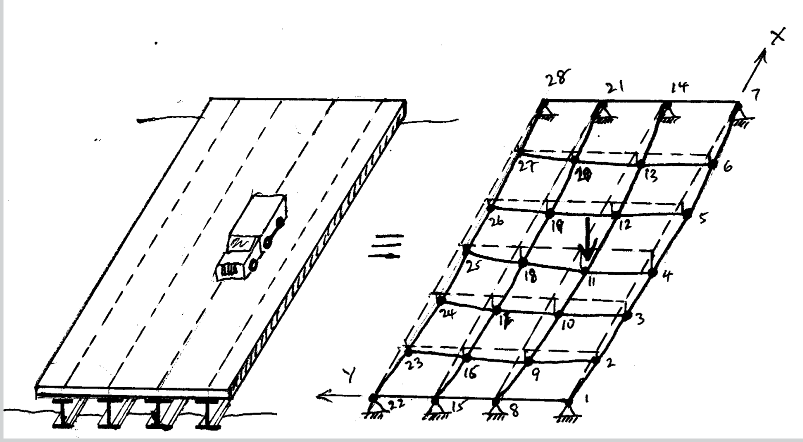
Grillage Analogy Method

Procedure

- When a bridge deck is analyzed by the method of Grillage Analogy, there are essentially five steps to be followed for obtaining design responses :
  - Idealization of physical deck into equivalent grillage
  - Evaluation of equivalent elastic inertia of members of grillage
  - Application and transfer of loads to various nodes of grillage
  - Determination of force responses and design envelopes and
  - Interpretation of results.

Grillage Analogy Method

Grillage of Beam and Slab Deck



Bridge Deck Idealized Model (Deflected)

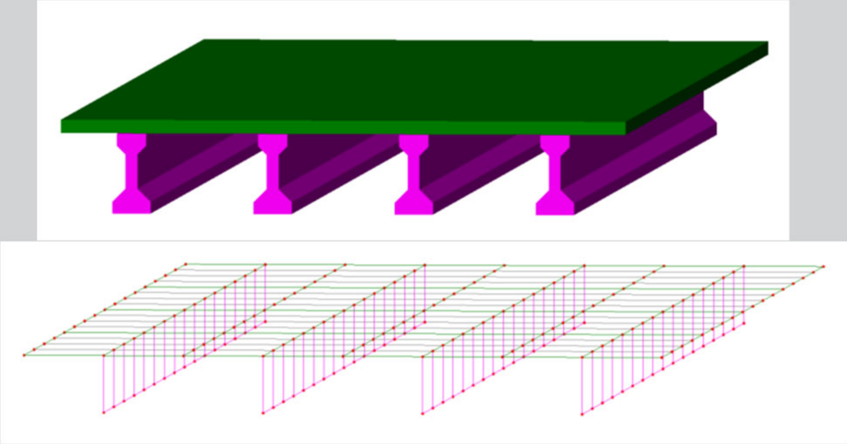
Plates with Eccentric Beams (PEB)

Refinement to Grillage Method

- Explicitly models the deck slab and girders separately - composite behavior no longer needs to be approximated.
- The depth of the structure is accounted for by locating the deck slab “plate” and the longitudinal girder “beams” at their respective centroids.
- The stiffness properties of the model is improved.
- Enforces compatibility between girders and allows the transfer of longitudinal shear forces between girders.
- *PEB is the recommended refined analysis tool for routine design of typical beam & slab bridges.*

Plates with Eccentric Beams

Illustration:



Plates with Eccentric Beams

Elements and Geometry

- Girders and the deck are connected at nodal locations. At a minimum, nodes should be located at the tenth points of each span.
- Deck and the girders/cross-members are modeled eccentric to each other.
- The number of longitudinal elements should be such that a reasonable aspect ratio is obtained. Up to 5 to 1 is acceptable though approaching 1 to 1 is best.
- At least two shell elements between each line of girders in order to capture shear lag behavior.
- By defining offsets or rigid links between the slab and the girder, the composite action is automatically achieved.

Plates with Eccentric Beams

Geometric Attributes – Beam & Slab deck

St Venant Torsion Constant (w/o warping)

The torsion constant , C of the beam can be given as:

Beam

- Sum of torsion constant s for web and flanges;
- For thin section such as b > 5t, C can be given as
  - $C = bt^3/3$ b = width, t = thickness

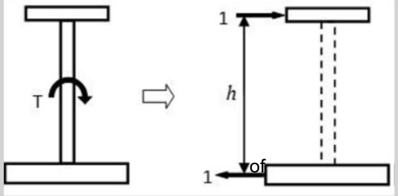
Slab

- Shell element – only thickness needs to be defined.

Plates with Eccentric Beams

Geometric Attributes – Beam & Slab deck

Equivalent Torsion Stiffness of I-Sections (considering warping)



The total equivalent torsional stiffness Jeq estimated as the calculated Saint-Venant stiffness J plus Jadditional:

$$J_{eq} = J + \frac{L_b h^2}{2G(\Delta_1 + \Delta_2)}$$

Lb = Length between cross-frames (in)
h = Distance between flange centroids (in)
G = Shear modulus of elasticity = *E*/[2×(1+n)] (ksi)
E = Modulus of elasticity (ksi)
n = Poisson's ratio
D1, D2 = Deflection of top and bottom flanges, respectively (in)

Plates with Eccentric Beams

Geometric Attributes – Beam & Slab deck

Equivalent Torsion Stiffness of I-Sections (considering warping)

- For an internal panel with equal size flanges, the equation will be:

$$J_{eq} = J + \frac{6EIh^2}{GL_b^2}$$

- For an end panel and with equal flanges the equation becomes:

$$J_{eq} = J + \frac{3EIh^2}{2GL_b^2}$$

where: I = Moment of inertia of a flange about a vertical axis (in4)

Plates with Eccentric Beams

Geometric Attributes – Diaphragms

Transverse Effects w/Diaphragms:

- For continuous cross frames or diaphragms, transverse load distribution is significant and needs to be simulated in the PEB.
- If the diaphragm(s) are discontinuous, the transverse effects are insignificant and can be ignored.
- Provide bracing to the girders during erection (stability).
- Main load path for curved girder bridges.

Plates with Eccentric Beams

Geometric Attributes – Cross Frame/ Diaphragm

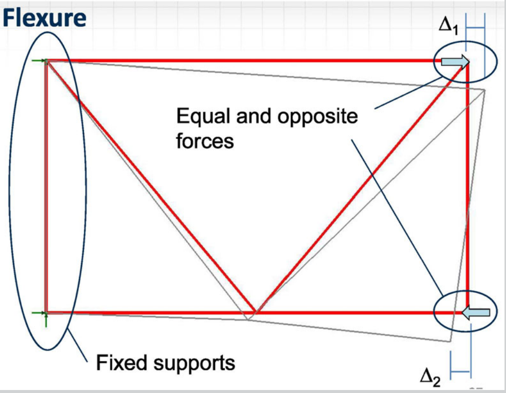
Equivalent Stiffness of Cross frame/Diaphragm:

- Typical cross frame or diaphragm can be a plate diaphragm, an X-, K-, or inverted K-type or others.
- Several approaches are commonly taken to model the stiffness of a cross frame or diaphragm in a 2D analysis.
- In most cases, differential rotation (twisting) of adjacent girders will engage the flexural stiffness of the cross frames.
- Equivalent stiffness approach taking shear deformation into account is recommended (Timoshenko formulation).

Plates with Eccentric Beams

Geometric Attributes – Diaphragm

Equivalent Flexural Stiffness of Diaphragm



- Modeling as a cantilever
- Apply unit force couple at top & bot
- Determines $\Delta 1$ and $\Delta 2$
- $\theta = ML/EI_e$ & $\theta = (\Delta 1+\Delta 2)/depth$

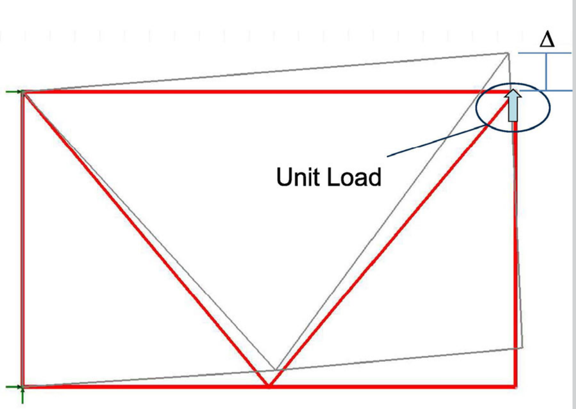
$\Rightarrow I_e = ML/E\theta$

I_e becomes the moment of inertia of the diaphragm modelled as a line element in the PEB.

Plates with Eccentric Beams

Geometric Attributes – Diaphragm

Equivalent Shear Stiffness of Diaphragm



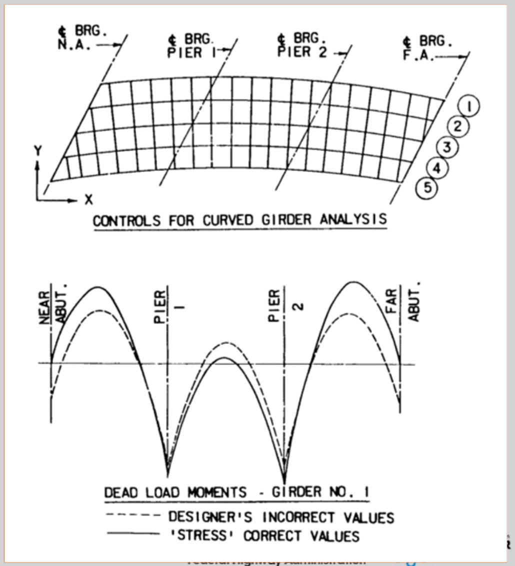
$\Delta =$ $VL^3/3EIeq$ + $PL/GAeq$

$$A_{eq} = \frac{VL}{G\left(\Delta - \frac{VL^3}{3EI_{eq}}\right)}$$

Aeq = equivalent shear area
D = deflection due to unit force
V = unit force
L = length between girders
E = steel modulus of elasticity
Ieq = equivalent moment of inertia
G = shear modulus of elasticity = E/[2×(1+n)]
n = Poisson’s ratio

Boundary Conditions

- Sufficient boundary conditions needed to ensure stability
- Commonly idealized with full fixity to either translation or rotation
- Friction or non-linear behavior for advanced analyses
- Consequences of incorrect boundary conditions

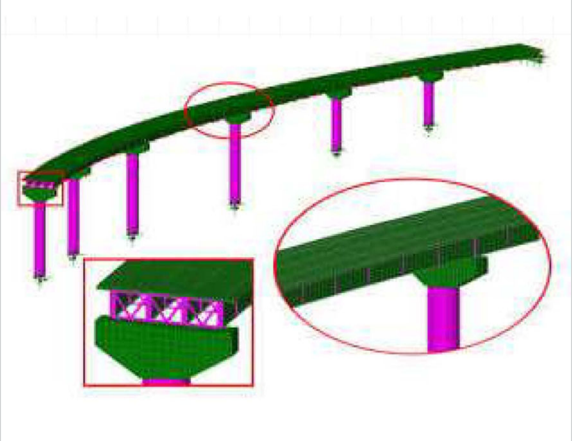


Bearings

- Never friction free or perfectly rigid
- Restrained directions often have small movement before engaging restraints
- Very large forces can develop in model that don't occur in real bearing
- Directions of restraint and release often a source of trouble (modeling bearing orientation)
- Eccentricity of bearing from neutral axis – rotation/movement coupling

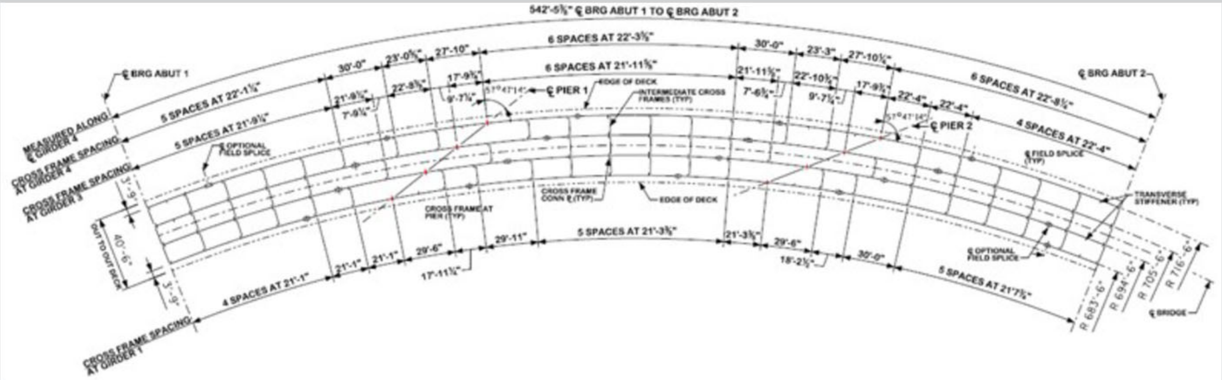
Rotation/Movement Coupling

- When girders rotate at support, bearings move longitudinally
- Different rotations among girders means different longitudinal movements
- Can result in large, self-equilibrating longitudinal forces in bearings at a support



Class Exercise

Using what we learned from 2-D, what is the best way to mesh the following curved structure?



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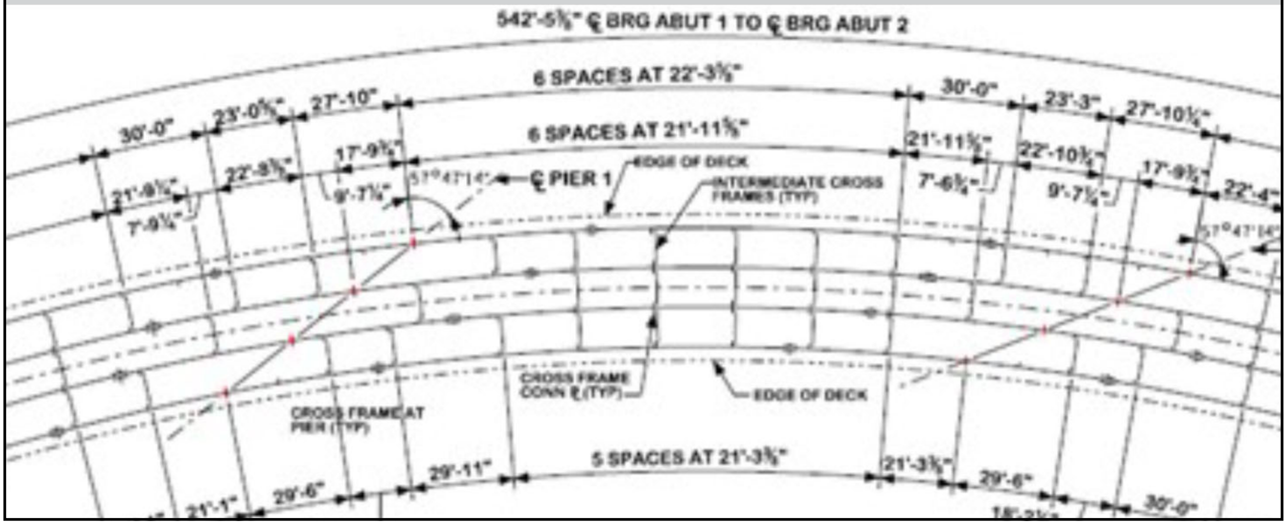
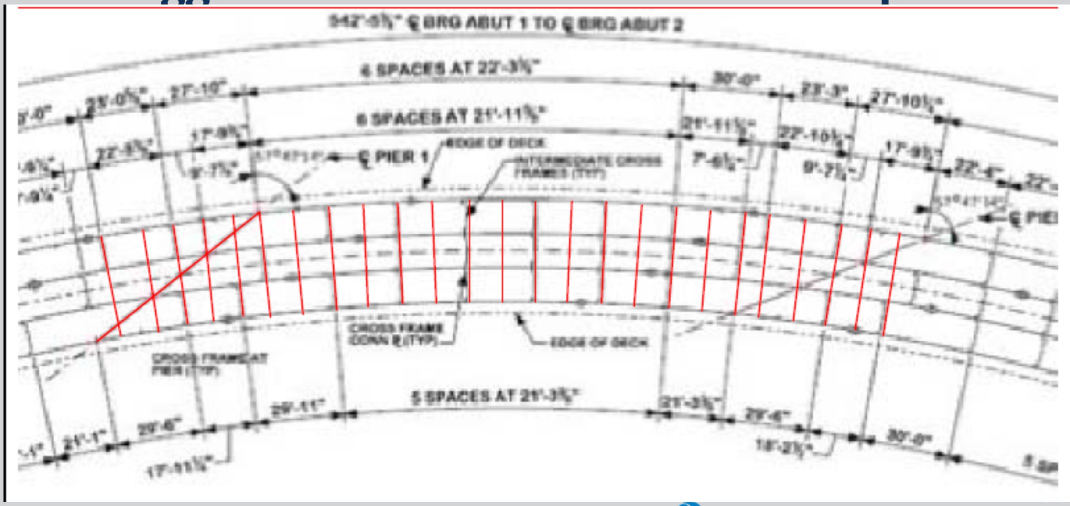
Class Exercise

Using what we learned from 2-D, what is the best way to mesh the following curved structure?



Class Exercise

Suggested 2-D Mesh of the Center Span



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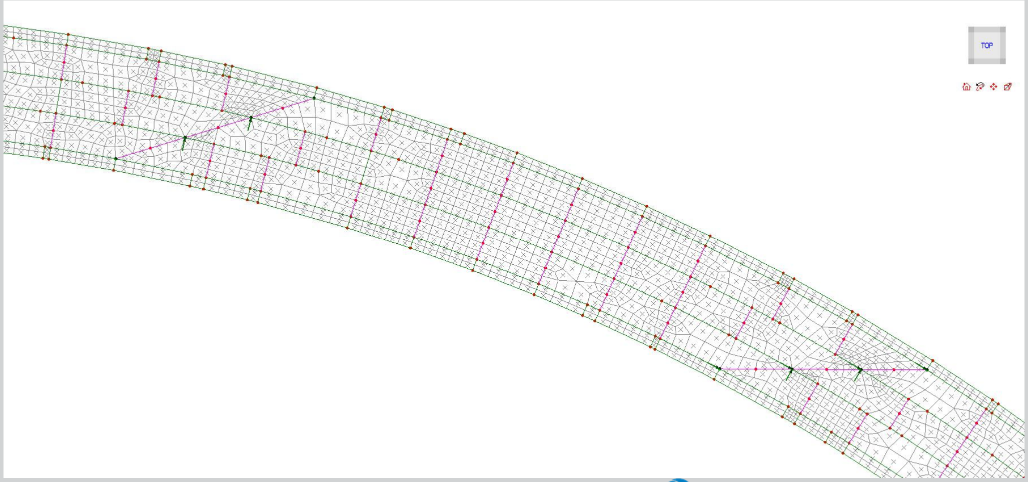
Class Exercise

FEA software generated Mesh of the Center Span



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Three Dimensional Analysis

Element types & usage:

- Many more element types are available compared to 1 D & 2D
- When mixing multiple element types, compatible element DOF need to be ensured at common nodes.
  - For instance, connecting a beam element with rotational stiffness to a solid element with only translational stiffness will not result in moments being transferred across the joint.

Three Dimensional Analysis

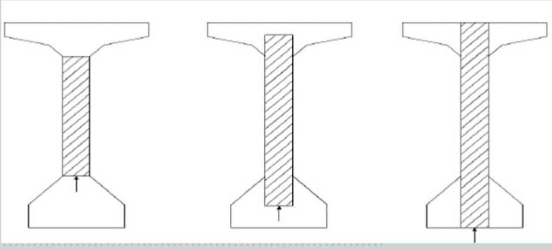
ELEMENT TYPE	TYPICAL APPLICATIONS
BAR (TRUSS)	Steel cross-frame diagonals on a slab girder structure.
BEAM	Steel cross-frame top and bottom chords, girder flanges, diaphragm flanges, diaphragms, longitudinal and transverse stiffeners, shear connectors. etc
SURFACE (SHELL)	Concrete deck slabs, girder flanges and webs, and plate diaphragms, stiffeners and diaphragms.

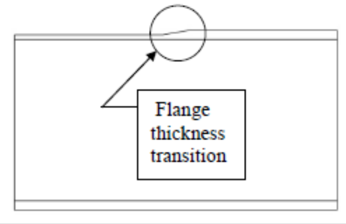
Three Dimensional Analysis

ELEMENT TYPE	TYPICAL APPLICATIONS
VOLUME	Though full 3-D stress field can be captured, it is rarely used.
CONSTRAINTS AND RIGID LINKS	Modeling composite action between girders and deck slabs, offset between the centroid and the surface of an element & modeling elements that are very rigid compared to surrounding elements, such as integral concrete bent caps.
SPRING & POINT ELEMENTS	Used at interfaces or boundary conditions, e.g. bearings, substructure/foundation stiffnesses.

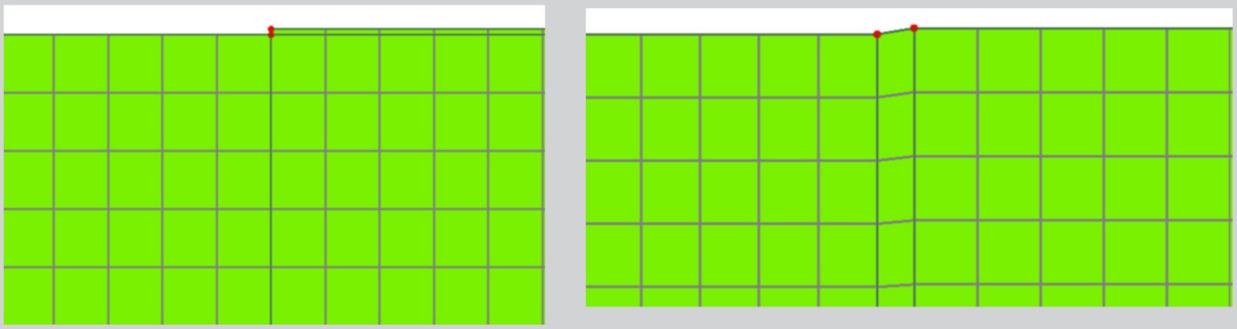
3D Geometry

- Actual vs. assumed geometry
- Recommended modeling techniques





Flange thickness transition modeling



(a) Poor

(b) Good

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Three Dimensional Analysis

Number of Elements:

- The number of elements required is a balance between accuracy and efficiency.
- Varying the mesh size over a given model can also achieve efficiency.
- *As a rule of thumb, if increasing the number of elements results in a difference of less than five percent, the coarser mesh is sufficient*.
- The degrees of freedom of the elements being connected have to be compatible.

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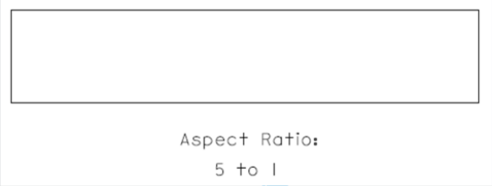
Three Dimensional Analysis

Aspect Ratio & Geometric Distortion:

- LRFD sets 5:1 as the maximum aspect ratio limit. And the interior angles of elements should be equal to the ideal angle plus or minus 60 percent.
- Depending on the element formulation, the application, and the solver being used, larger aspect ratios and geometric distortion may be acceptable, but as always, the results should be verified.



Three Dimensional Analysis

Girder Flanges:

- *AASHTO Article C4.6.3.3.1* recommends a minimum of 5, and preferably at least nine nodes per beam span.
- Modeling the flanges using beam elements is recommended for I-shaped girders. If shell elements are used, at least two elements are needed to define the flange width for I-girders, one on each side of the web centerline. For box girders, at least two elements are needed and four are recommended to capture shear lag effects.

Three Dimensional Analysis

Girder Webs:

- Number of shell elements required can vary from 1 to 12.
- If capturing the parabolic shear behavior is important, use at least four elements throughout the depth of the web.
- Number of elements required often depends on the number of elements in the flanges.
- Locations of longitudinal and vertical stiffeners may be a consideration in the location and number of elements in the web.

Polling Question 1 (L2S43Q1):

When carrying out the grillage method, skew mesh is not recommended when the skew angle is more than:

a) 15 deg

b) 20 deg

c) 30 deg

d) 45 deg

Polling Question 2 (L2S44Q2):

When mixing multiple element types in 2D or 3D analysis, compatible element DOF needs to be ensured at common nodes.

a) True

b) False

Polling Question 3(L2S45Q3):

If capturing the parabolic shear behavior is important, at least how many elements are recommended throughout the depth of the web?

a) 2

b) 3

c) 4

d) 5

Learning Outcome Review

- Explain the general steps and key parameters that affect building a good general bridge FEA model.
- Describe the basic procedures for 1D, 2D and 3D refined analysis.

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Next:
Lesson 3: Modeling Special Topics

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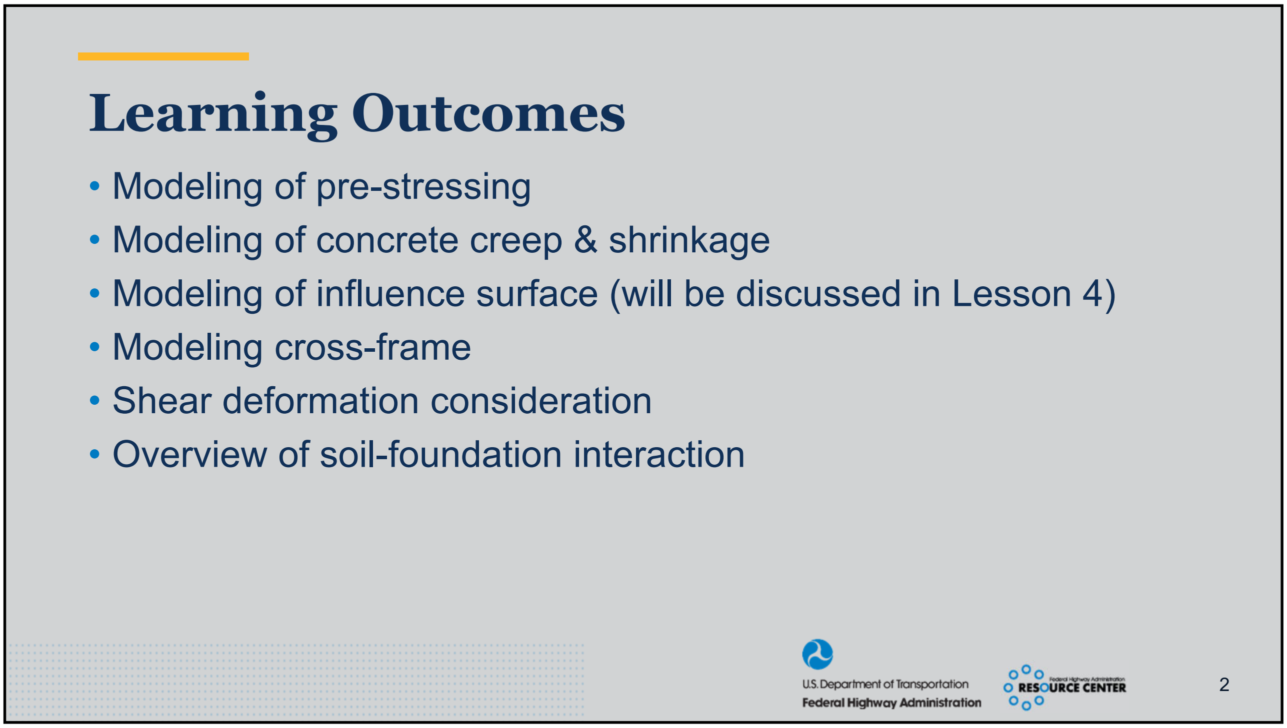
Lesson 3
Modeling Special Topics



Learning Outcomes

- Modeling of pre-stressing
- Modeling of concrete creep & shrinkage
- Modeling of influence surface (will be discussed in Lesson 4)
- Modeling cross-frame
- Shear deformation consideration
- Overview of soil-foundation interaction

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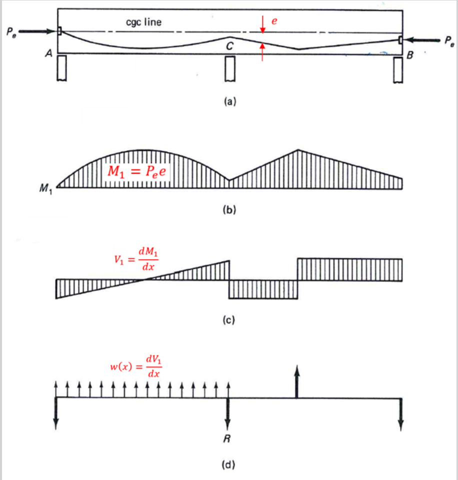
Modeling of pre-stressing (FHWA Manual 4.1.3)

- Equivalent element loads by classical $M - V - W$ diagrams
- Find equivalent segment loads
- Use equivalent truss element with fixed-end forces

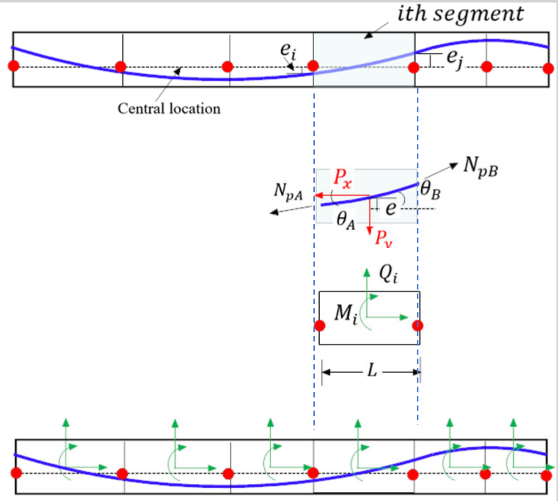
Equivalent element load method

- Draw classical $M - V - W$ diagrams:
  - $M_1 = P_e e$
  - $V_1 = \frac{dM_1}{dx}$
  - $w(x) = \frac{dV_1}{dx}$
- Apply $w(x)$ and R's to the girders. The corresponding moment is the moment due to prestress tendon.



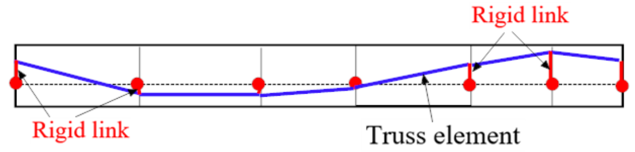
Equivalent segment loads

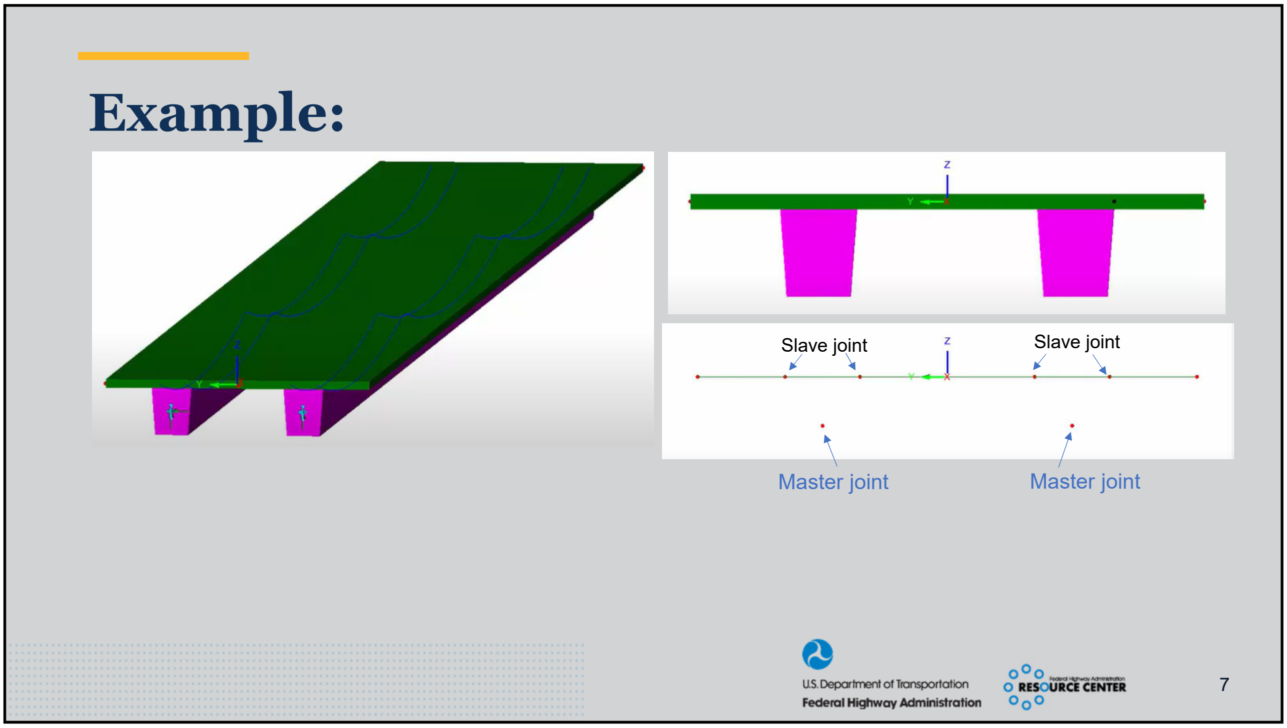
- Cut member into several segments
- For each end of segment, find segment end forces due to prestress force; find reactions P_x, P_y
  - $\sum F_x = 0;\ \sum F_y = 0:$
  - $P_x + N_{pA}cos\theta_A = N_{pB}cos\theta_B$ (1)
  - $P_y + N_{pA}sin\theta_A = N_{pB}sin\theta_B$ (2)
  - Solve P_x, P_y.
- The individual segment's equivalent forces are:
  - $M_i = P_x e;\ Q_i = P_y$



Equivalent truss element with fixed-end forces

- Treat tendon as truss element(s):
  - With consideration of truss fixed end forces at joints
  - With rigid link connecting truss to the centroid of segment





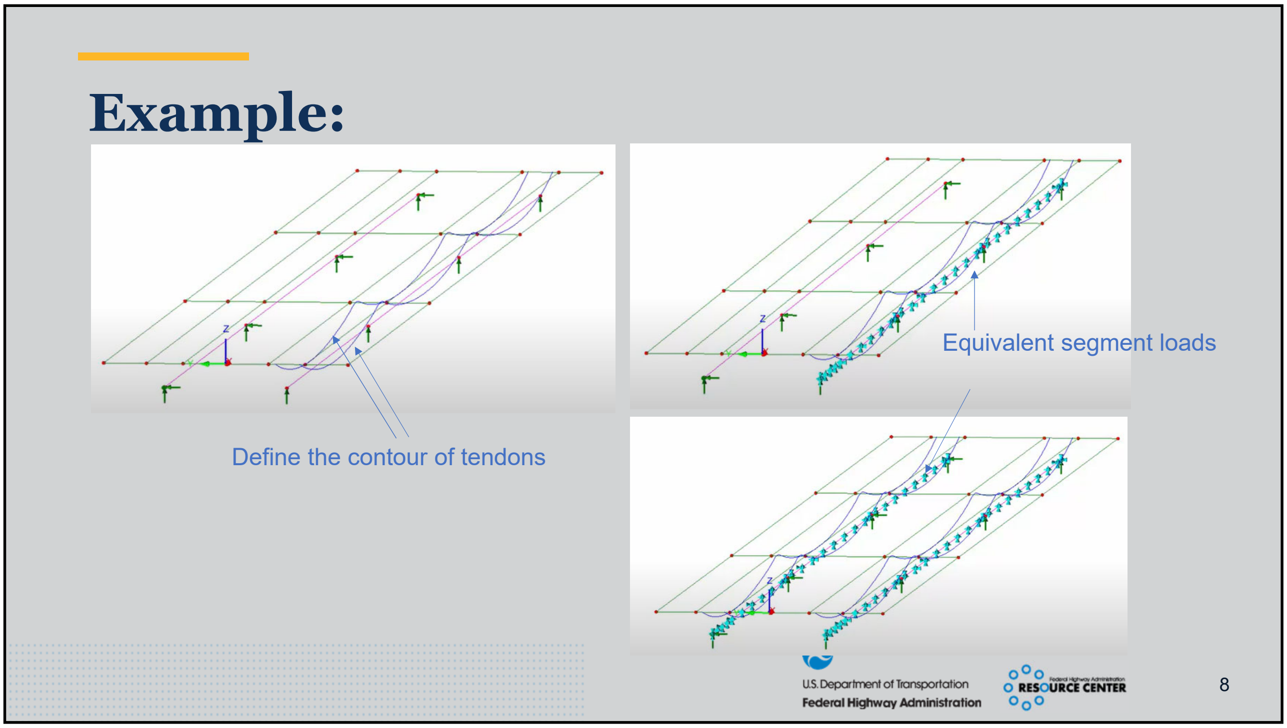
Example:

Slave joint

Slave joint

Master joint

Master joint



Example:

Define the contour of tendons

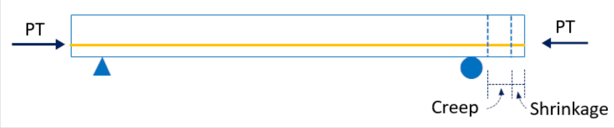
Equivalent segment loads

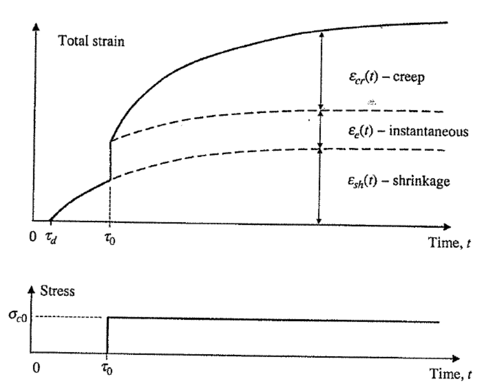
Modeling of concrete creep & shrinkage (FHWA Manual 4.1.2)

- Definition of creep & shrinkage
- Creep coefficient & creep function
- Shrinkage coefficient
- Cross section analysis
- Matrix method for creep & shrinkage
- FEM for creep & shrinkage

Definition of creep & shrinkage

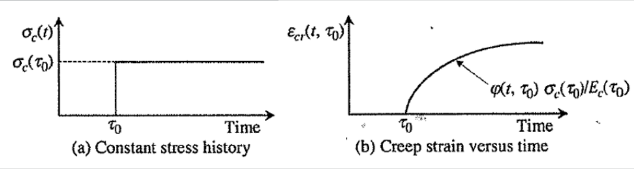
- Concrete will creep with time when subjected to a sustained stress from tendons
- Concrete will shrink with time





Definition of creep

- Creep strain, $\varepsilon_{cr}(t,\tau_0)$ = Concrete strain develops, when subjected to a sustained stress at time τ_0
- Creep coefficient, $\phi(t,\tau_0)$ = the ratio of creep strain to the instantaneous elastic strain
  - $\phi(t,\tau_0) = \frac{\varepsilon_{cr}(t,\tau_0)}{\varepsilon_e(\tau_0)}$
  - $\sigma_c(\tau_0) = E_c(\tau_0)\varepsilon_e(\tau_0)$

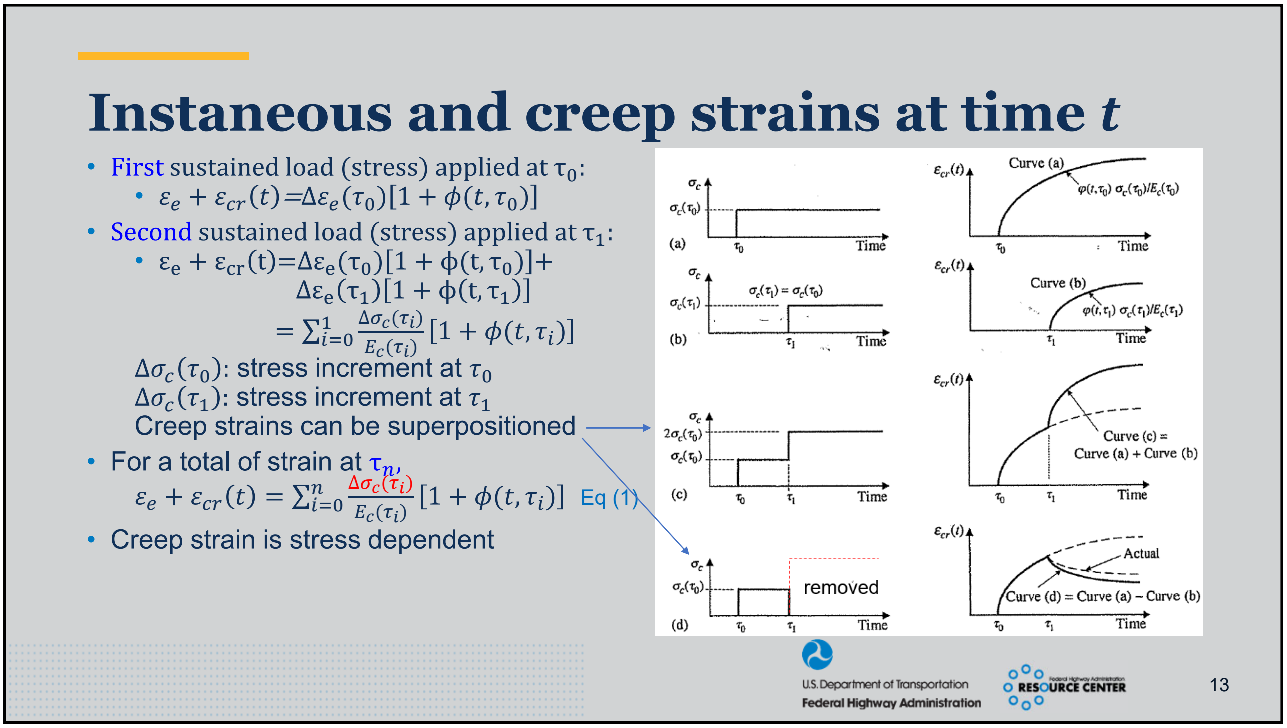


(a) Constant stress history
(b) Creep strain versus time

  - Note: $E_c(t) = \underbrace{\left(e^{0.38-\sqrt{\frac{28}{t}}}\right)^{0.5}}_{\cong 1} \times E_c\ (28\ days) \cong 85\% - 115\%\ of\ E_c\ at\ 28\ days$

Creep coefficient

- $\phi(t,\tau) = 1.9k_s k_{hc} k_f k_{td} \tau^{-0.118}$; (AASHTO LRFD Specs.):
  - τ = the stress first applied at concrete age τ
  - t = the creep considered at time t
  - k_s = coefficient related to the effect of the volume-to-surface ratio of the component (i.e. $f(\frac{V}{S})$)
  - k_f = factor for the effect of concrete strength (i.e. $f(f_c')$)
  - k_{hc} = humidity factor for creep (i.e. $f(H)$)
  - k_{td} = time development factor (i.e. $f(f_c'; t)$)



Instaneous and creep strains at time *t*

- First sustained load (stress) applied at τ_0:
  - $\varepsilon_e + \varepsilon_{cr}(t) = \Delta\varepsilon_e(\tau_0)[1 + \phi(t, \tau_0)]$
- Second sustained load (stress) applied at τ_1:
  - $\varepsilon_e + \varepsilon_{cr}(t) = \Delta\varepsilon_e(\tau_0)[1 + \phi(t, \tau_0)] + \Delta\varepsilon_e(\tau_1)[1 + \phi(t, \tau_1)]$

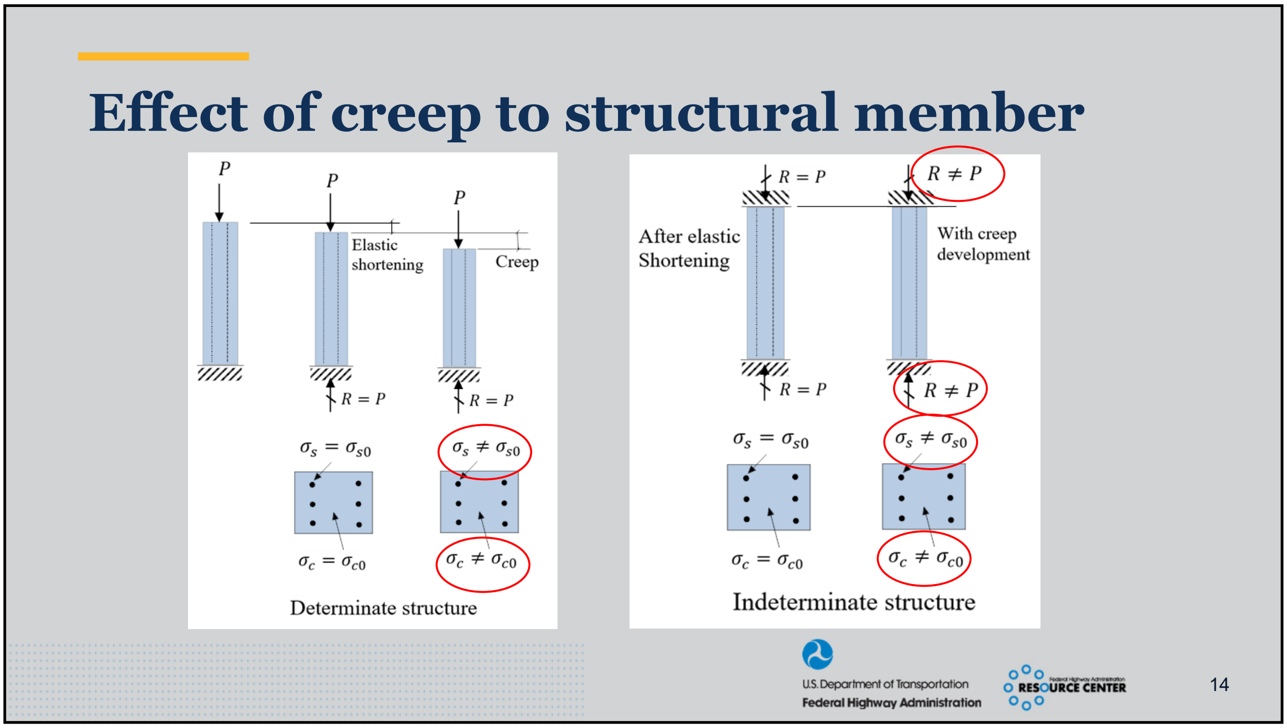
    $= \sum_{i=0}^{1} \frac{\Delta\sigma_c(\tau_i)}{E_c(\tau_i)}[1 + \phi(t, \tau_i)]$

  $\Delta\sigma_c(\tau_0)$: stress increment at τ_0

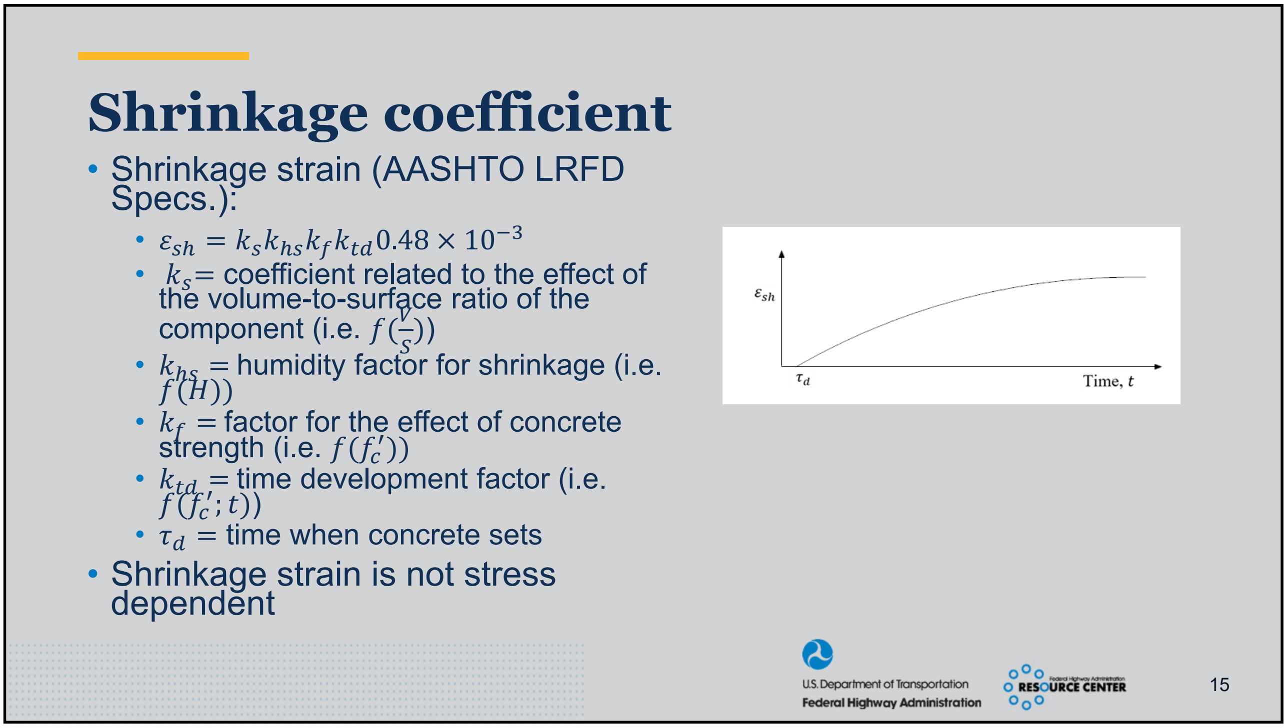
  $\Delta\sigma_c(\tau_1)$: stress increment at τ_1

  Creep strains can be superpositioned
- For a total of strain at τ_n,

  $\varepsilon_e + \varepsilon_{cr}(t) = \sum_{i=0}^{n} \frac{\Delta\sigma_c(\tau_i)}{E_c(\tau_i)}[1 + \phi(t, \tau_i)]$ Eq (1)
- Creep strain is stress dependent



Effect of creep to structural member



Shrinkage coefficient

- Shrinkage strain (AASHTO LRFD Specs.):
  - $\varepsilon_{sh} = k_s k_{hs} k_f k_{td} 0.48 \times 10^{-3}$
  - k_s= coefficient related to the effect of the volume-to-surface ratio of the component (i.e. $f(\frac{V}{S})$)
  - k_{hs} = humidity factor for shrinkage (i.e. $f(H)$)
  - k_f = factor for the effect of concrete strength (i.e. $f(f_c')$)
  - k_{td} = time development factor (i.e. $f(f_c'; t)$)
  - τ_d = time when concrete sets
- Shrinkage strain is not stress dependent

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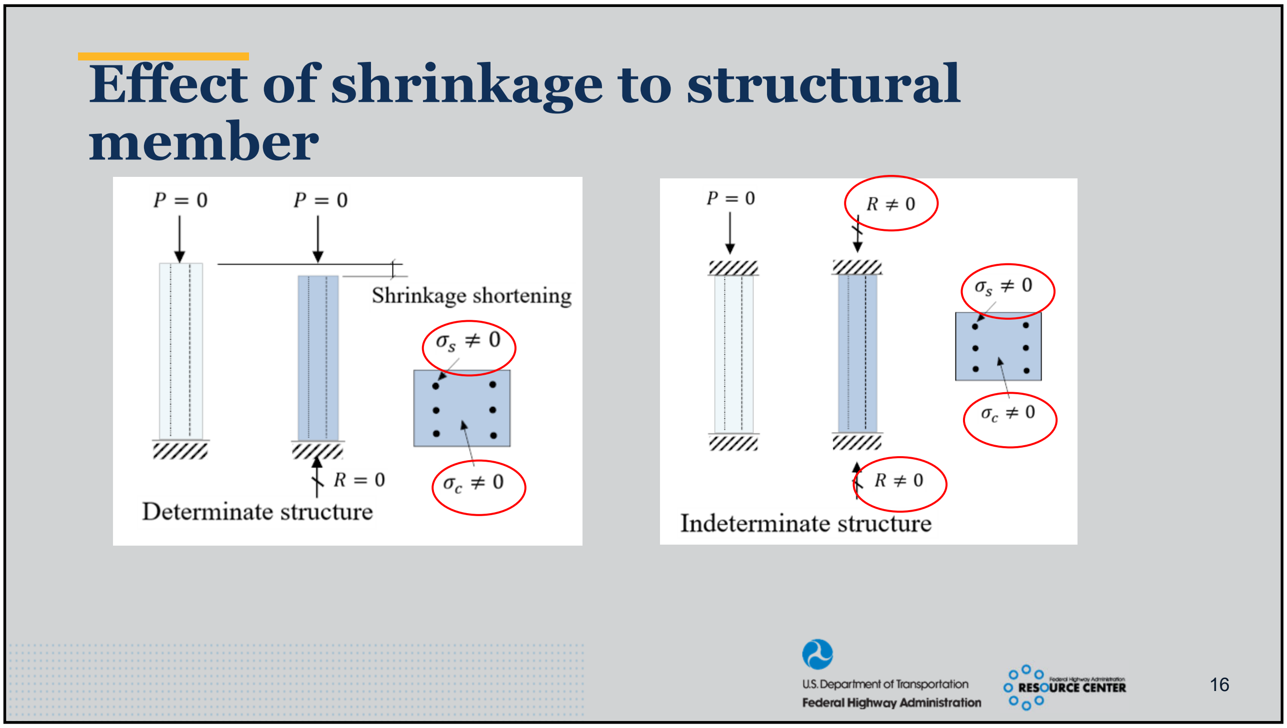

Effect of shrinkage to structural member

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Knowledge Review -- Question 1 (L3 S17 Q1)

The total creep strain can be obtained by superimposing the creep strains due to individual sustained loads (stresses).

A. True.

B. False.

Knowledge Review -- Question 2 (L3 S18 Q2)

The shrinkage strain is dependent on the sustained loads (stresses).

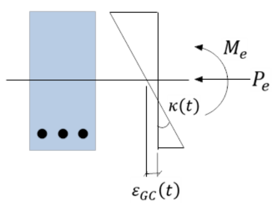
A. True.

B. False.

Conventional structural analysis for creep & shrinkage consideration

- STEP 1: Perform moment-curvature ($M - \kappa$) analyses at key cross sections first
  - $P_e = P_i = \int_A \sigma(t) dA$ (equilibrium)
  - $M_e = M_i = \int_A y\sigma(t) dA$ (equilibrium)
  - $\varepsilon(\tau_j) = \varepsilon_s(\tau_j)$ (compatibility)
  - Stress-strain relationship:
  - $\varepsilon(\tau_j) = \varepsilon_e(\tau_j) + \varepsilon_{cr}(\tau_j) + \varepsilon_{sh}(\tau_j)$

  $= \sum_{i=0}^{j} \frac{\Delta\sigma_c(\tau_i)}{E_c(\tau_i)}\left[1 + \phi(\tau_j, \tau_i)\right] + \varepsilon_{sh}(\tau_j)$ ✓
  - $\varepsilon(\tau_j) = \frac{\sigma_c(\tau_j)}{E_c(\tau_0)}$ for the conventional M- κ analysis ✗



Conventional structural analysis for creep & shrinkage consideration

- STEP 2: Find equivalent nodal forces
  - Based on cross-sectional ϵ and κ from Step 1
  - Get fixed-end forces
  - See Appendix A for details

FEM analysis for creep & shrinkage consideration

- The incremental equivalent nodal forces, $\{\Delta P\}$, at time step t_{i+1} can be obtained from
- Principle of virtual work (see Appendix A for details)
  - $\{\Delta P\} = \underbrace{\int [B]^T\{\sigma\}d\phi\, dv}_{\text{Contribution from creep}} + \underbrace{\int [B]^T[E]d\varepsilon_{sh}dv}_{\text{Contribution from shrinkage}}$
  - where,
    - $\{\sigma\}d\phi = \sum_{j=0}^{i}\{\Delta\sigma_j\}[\phi(t_{i+1},t_j) - \phi(t_i,t_j)]$
    - $d\varepsilon_{sh} = \varepsilon_{sh}(t_{i+1},\tau_d) - \varepsilon_{sh}(t_i,\tau_d)$
    - t_i = time at incremental step i; t_j = time when jth load applied
    - τ_d =time when concrete starts to set

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FEM analysis for creep & shrinkage consideration

- or at Gauss points (Use 2D element as an example):
  - $\{\Delta P\} = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}W_iW_j[B]_{ij}^{T}\{\sigma\}_{ij}d\phi_{ij}|J_{ij}|$

    Contribution from creep

    $+ \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}W_iW_j[B]_{ij}^{T}[E]d\varepsilon_{sh}|J_{ij}|$

    Contribution from shrinkage

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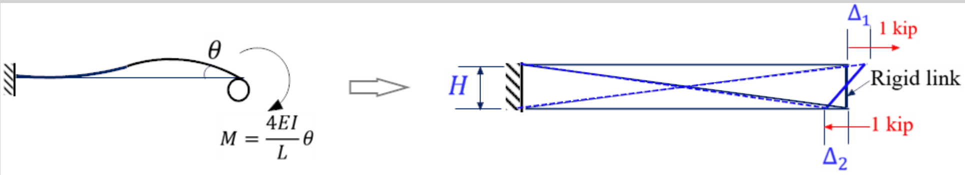
Simplified analysis of cross-frame (FHWA Manual 3.5.3)

- Simply cross-frame to a prismatic member
  - Effective moment of initial of prismatic member for bending, I_b

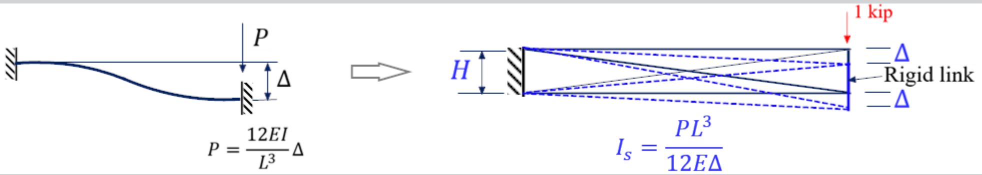


$$M = 1 \times H;\ \theta = \frac{\Delta_1 + \Delta_2}{H}$$

$$I_b = \frac{ML}{4E\theta}$$

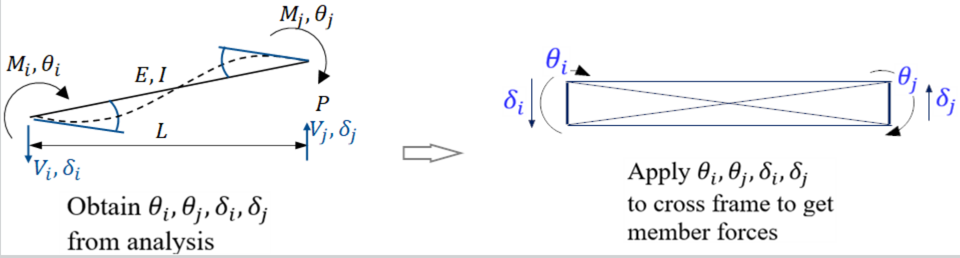
Simplified analysis of cross-frame

- Simply cross-frame to a prismatic member
  - Effective moment of initial of prismatic member for shear, I_s



Design of cross-frame members

- Obtain $\theta_i, \theta_j, \delta_i, \delta_j$ from structural analysis
- Apply $\theta_i, \theta_j, \delta_i, \delta_j$ to cross frame to get member forces
- Design members per AASHTO

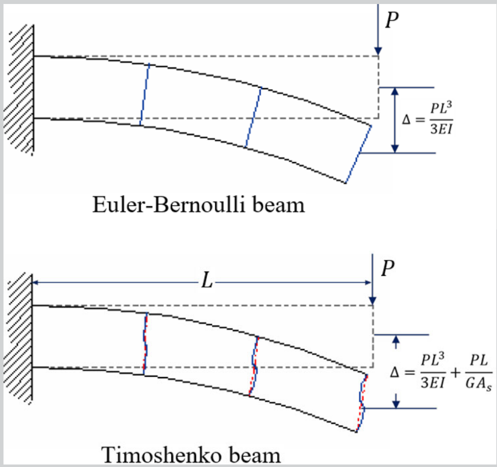


Obtain $\theta_i, \theta_j, \delta_i, \delta_j$ from analysis

Apply $\theta_i, \theta_j, \delta_i, \delta_j$ to cross frame to get member forces

Consider shear deformation (FHWA Manual 2.3.4)

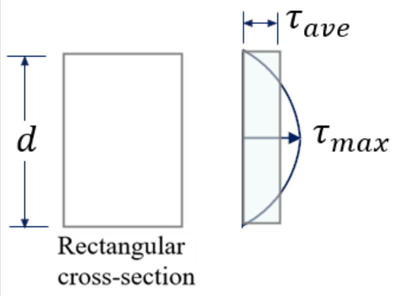
- Euler-Bernoulli beam
  - Plane sections remaining plane after bending
  - $\Delta = \frac{PL^3}{3EI}$
- Timoshenko beam
  - Plane of the section do not remain in the plane after bending
  - $\Delta = \frac{PL^3}{3EI} + \frac{PL}{GA_s}$ (More flexible)
    where A_s (shear area)



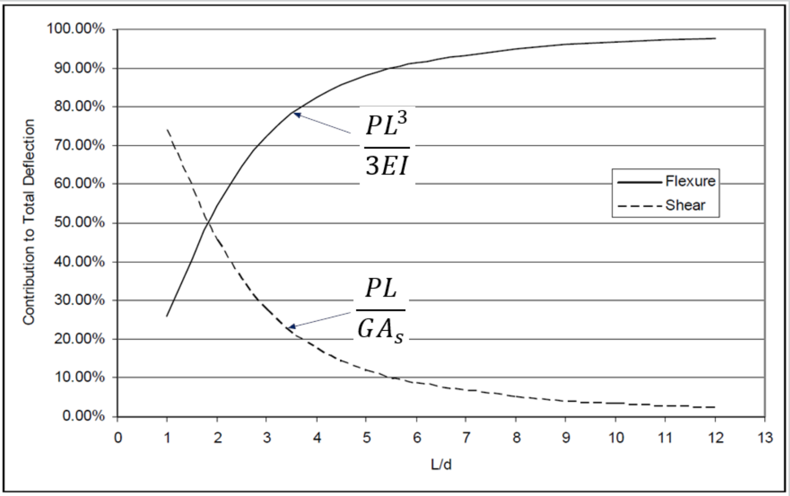
Definition of shear deformation

- Shear area, A_s -
  - $A_s = kA$
  - k = shear coefficient

$$= \frac{\tau_{ave}}{\tau_{max}}$$



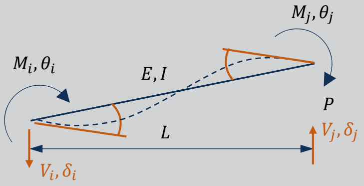
Shear deformation



Member stiffness matrix

- W/O shear deformation:

- $$\begin{Bmatrix} M_i\\ M_j\\ V_i\\ V_j\end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{-6EI}{L^2}\\ & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{-6EI}{L^2}\\ & symm. & \frac{12EI}{L^3} & \frac{12EI}{L^3}\\ & & & \frac{12EI}{L^3}\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_j\\ \delta_i\\ \delta_j\end{Bmatrix}$$

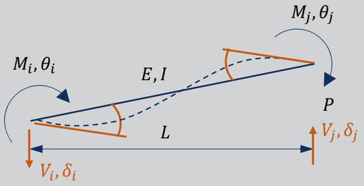


Member stiffness matrix

- With shear deformation:

- $$\begin{Bmatrix} M_i\\ M_j\\ V_i\\ V_j\end{Bmatrix} = \frac{EI}{(1+\phi)L}\begin{bmatrix} 4(1+\frac{\phi}{4}) & 2(1-\frac{\phi}{2}) & \frac{-6}{L} & \frac{-6}{L}\\ & 4(1+\frac{\phi}{4}) & \frac{-6}{L} & \frac{-6}{L}\\ & symm. & \frac{12}{L^2} & \frac{12}{L^2}\\ & & & \frac{12}{L^2}\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_j\\ \delta_i\\ \delta_j\end{Bmatrix}$$

- Where $\phi = \frac{12EI}{GA_sL^2}$

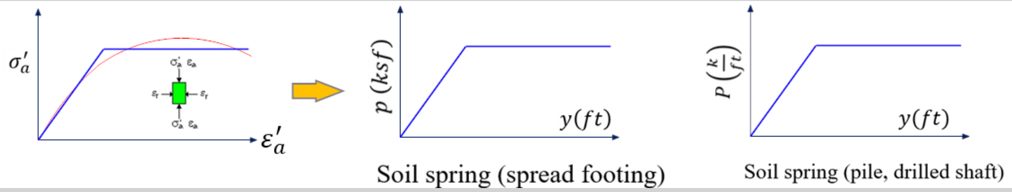


Soil-Foundation Interaction (FHWA Manual 5.2)

- Soil behavior under loading
- Modeling stiffness matrix for spread footing
- Modeling stiffness matrix for pile (drilled shaft) footing
- Modeling stiffness matrix for abutment

Soil behavior under loading

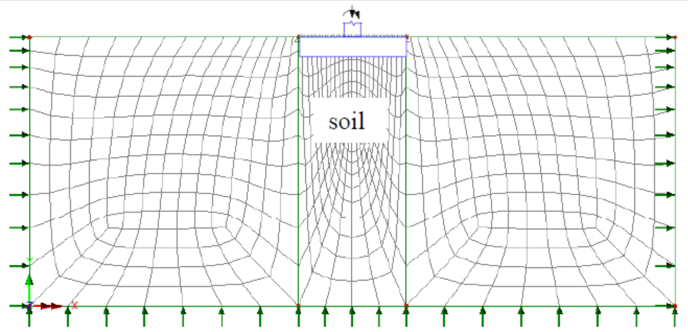
- $\sigma - \epsilon$ curve → soil spring constant



- Consult geotechnical engineer for using adequate soil spring model(s) for your project

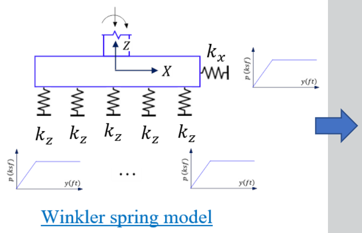
Modeling stiffness matrix for spread footing

- Direct approach:
  - Last resort

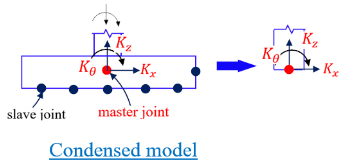


Modeling stiffness matrix for spread footing

- Substructuring approach:

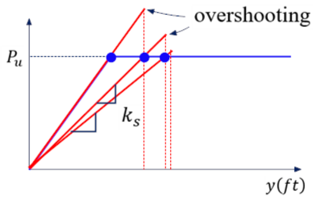


Winkler spring model



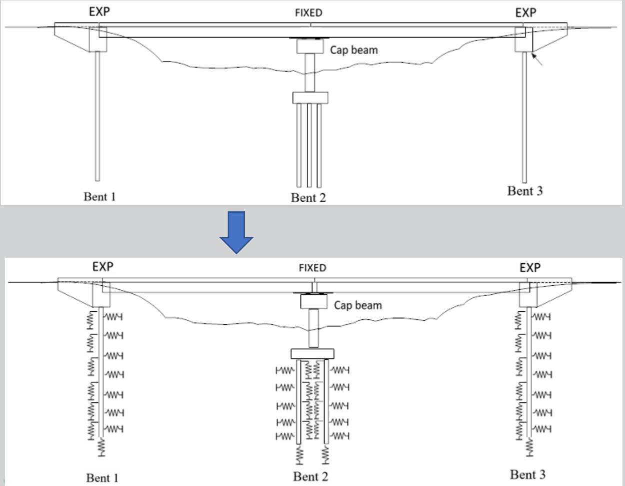
Condensed model

- Analysis: use equivalent (secant) stiffness if overshooting occurred.



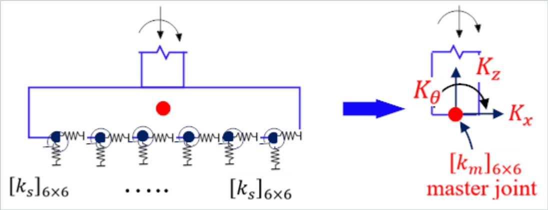
Modeling stiffness matrix for pile footing

- Direct approach:
  - Last resort

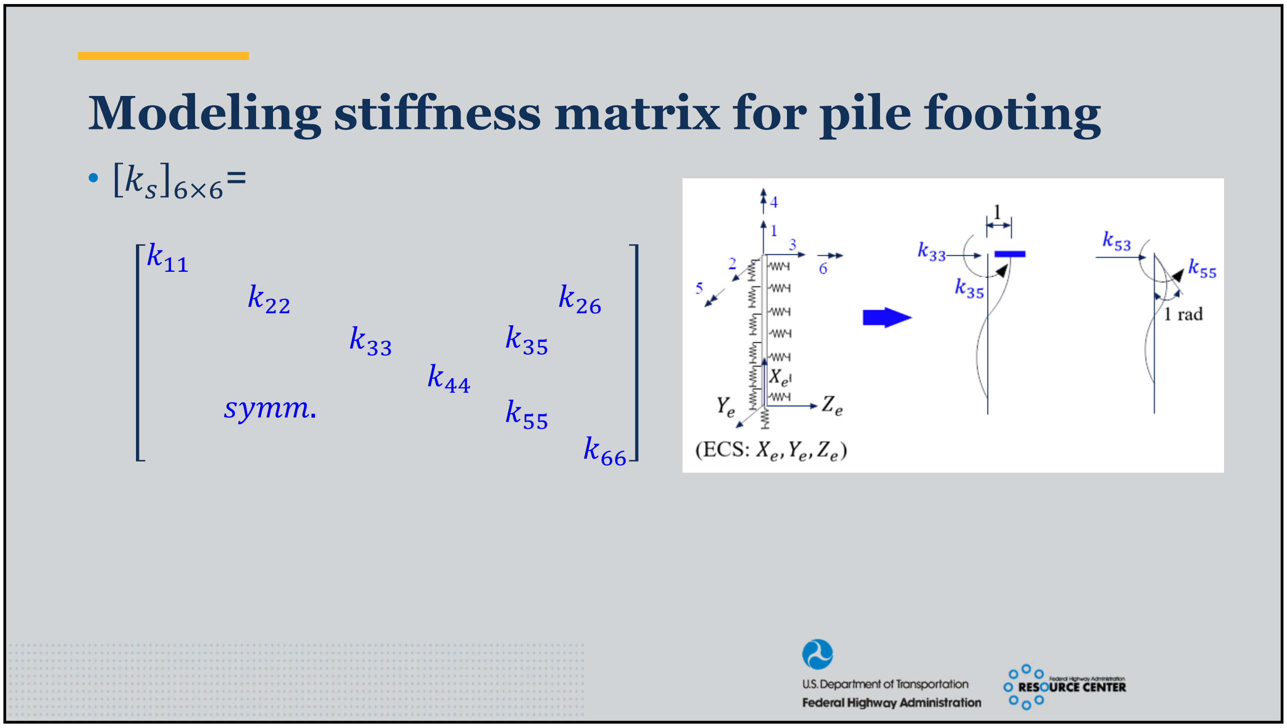


Modeling stiffness matrix for pile footing

- Substructuring approach:



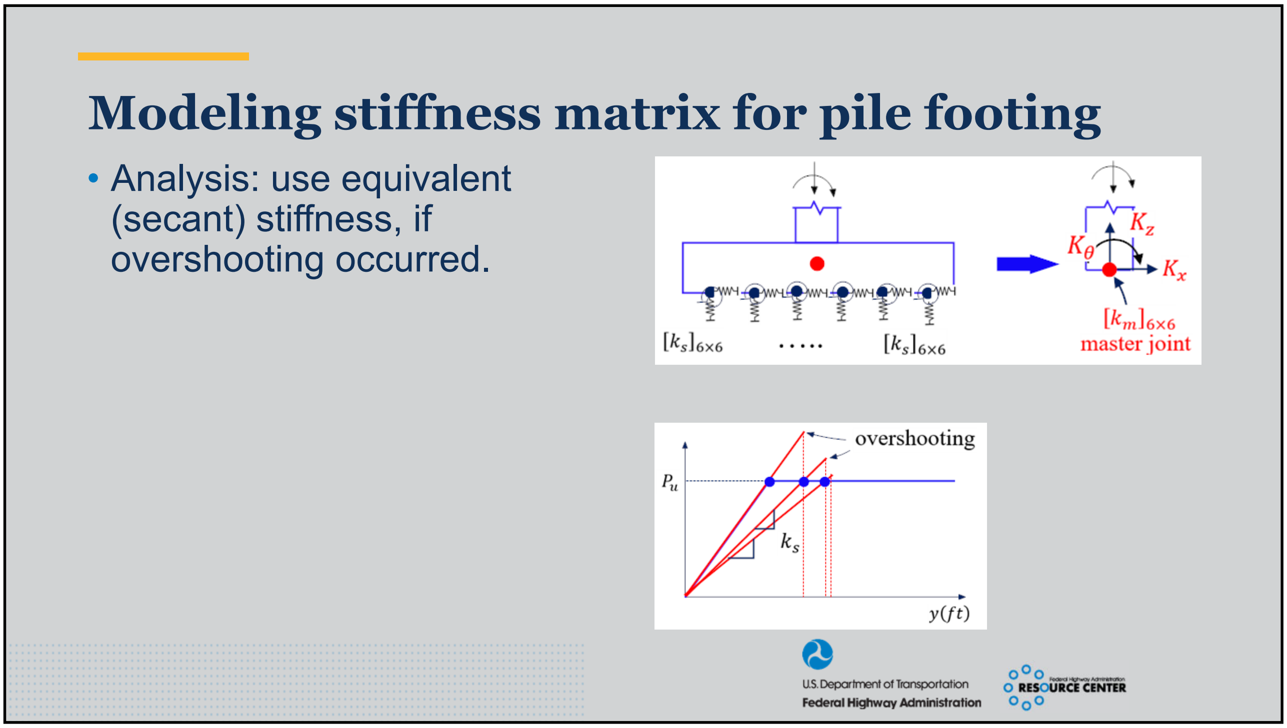
- Pile $[k_s]_{6\times6}$ can be obtained by any pile analysis software
  - LPILE, COM624, FB-Pier, etc.



Modeling stiffness matrix for pile footing

- $[k_s]_{6\times6}$=

$$\begin{bmatrix} k_{11} & & & & & \\ & k_{22} & & & & k_{26} \\ & & k_{33} & & k_{35} & \\ & & & k_{44} & & \\ & symm. & & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

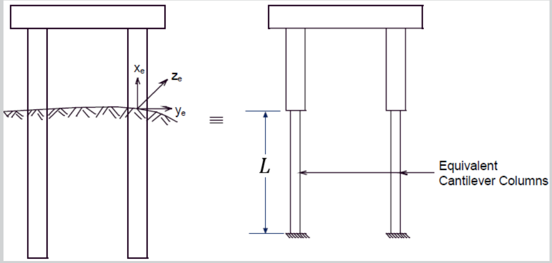


Modeling stiffness matrix for pile footing

- Analysis: use equivalent (secant) stiffness, if overshooting occurred.

Modeling stiffness matrix for pile footing

- Analysis: use equivalent cantilever column approach, if software does not provide point element option.



Equivalent cantilever column approach

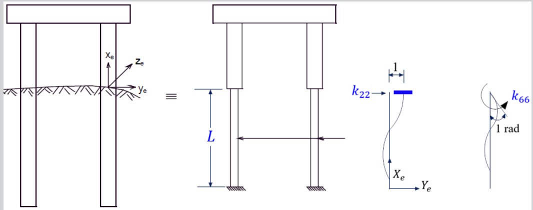
- Try to match $[k]_{equiv}$ to $[k_s]_{6\times6}$

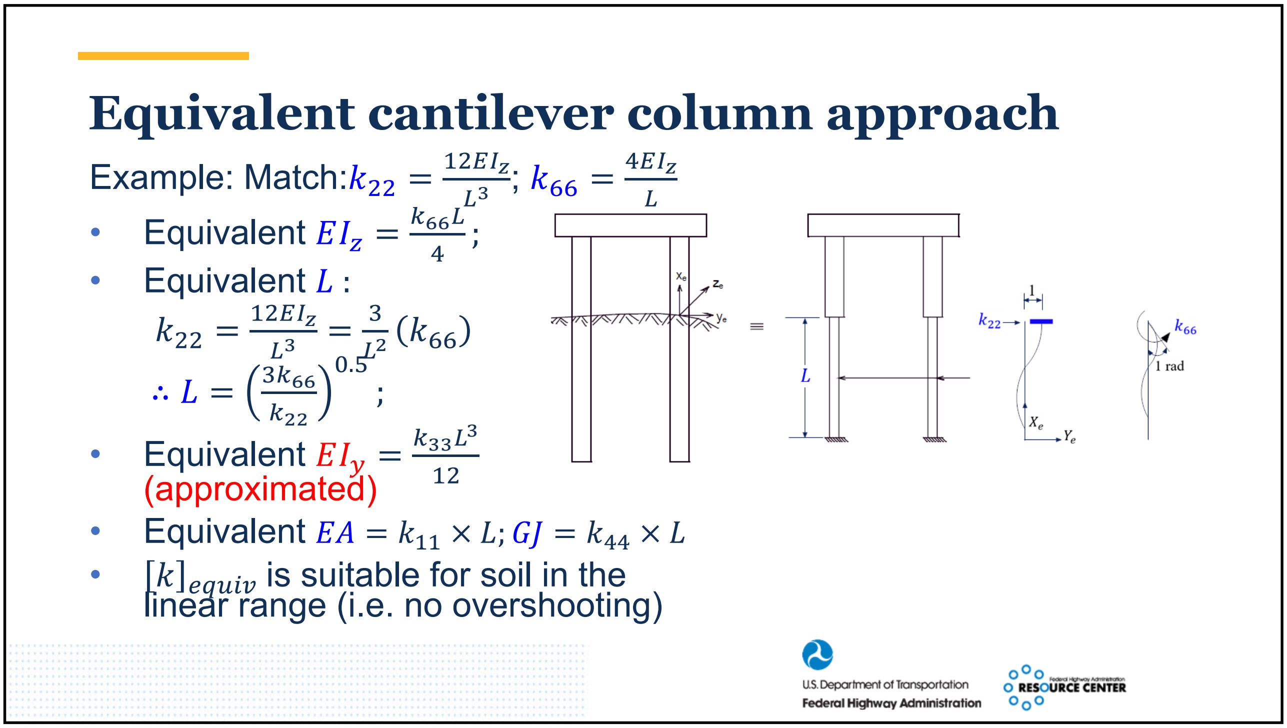
$$[k]_{equiv} = \begin{bmatrix} k_x = \frac{AE}{L} & & & & & \\ & k_y = \frac{12EI_z}{L^3} & & & & k_{y\theta} = \frac{-6EI_z}{L^2} \\ & & k_z = \frac{12EI_y}{L^3} & & k_{z\theta} = \frac{6EI_y}{L^2} & \\ & & & k_{\theta x} = \frac{GJ}{L} & & \\ & & & & k_{\theta y} = \frac{4EI_y}{L} & \\ & & & & & k_{\theta z} = \frac{4EI_z}{L} \end{bmatrix}$$

Equivalent cantilever column

$$[k_s]_{6\times6} = \begin{bmatrix} k_{11} & & & & & \\ & k_{22} & & & & k_{26} \\ & & k_{33} & & k_{35} & \\ & & & k_{44} & & \\ & symm. & & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

From soil analysis program





Equivalent cantilever column approach

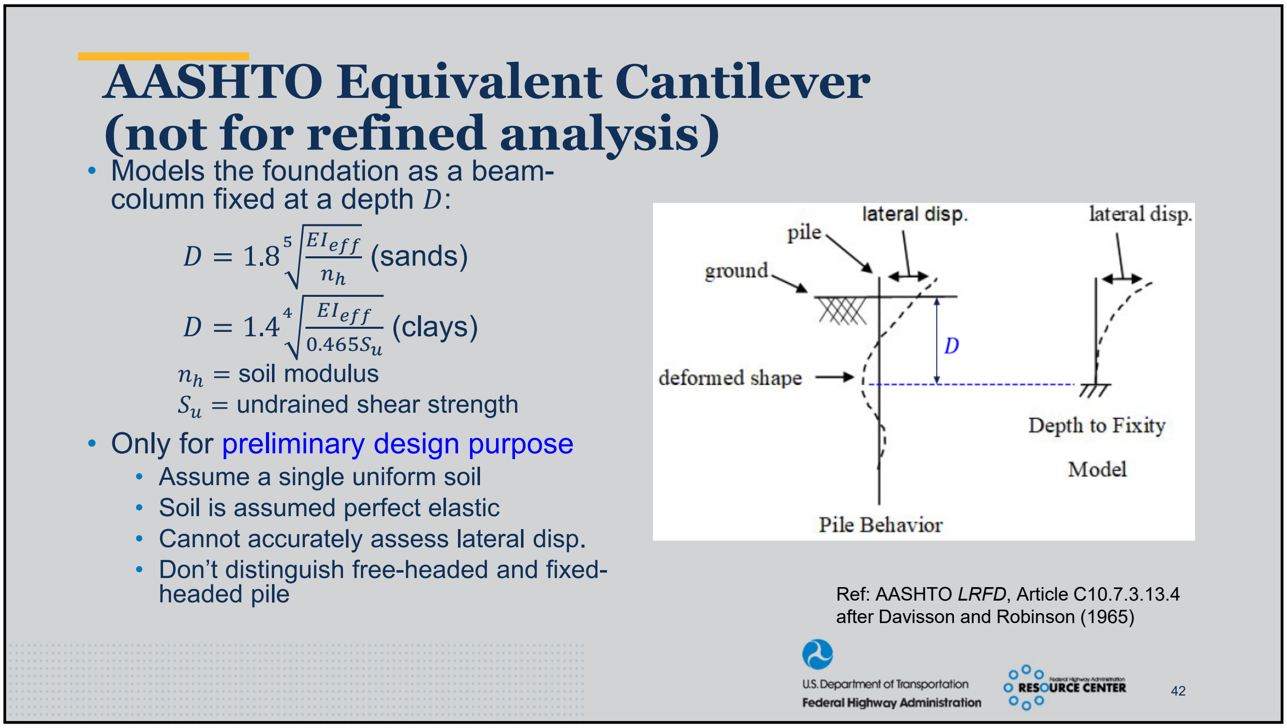
Example: Match: $k_{22} = \frac{12EI_z}{L^3}$; $k_{66} = \frac{4EI_z}{L}$

- Equivalent $EI_z = \frac{k_{66}L}{4}$;
- Equivalent L :

$$k_{22} = \frac{12EI_z}{L^3} = \frac{3}{L^2}(k_{66})$$

$$\therefore L = \left(\frac{3k_{66}}{k_{22}}\right)^{0.5};$$

- Equivalent $EI_y = \frac{k_{33}L^3}{12}$ (approximated)
- Equivalent $EA = k_{11} \times L; GJ = k_{44} \times L$
- $[k]_{equiv}$ is suitable for soil in the linear range (i.e. no overshooting)



AASHTO Equivalent Cantilever (not for refined analysis)

- Models the foundation as a beam-column fixed at a depth D:

$$D = 1.8\sqrt[5]{\frac{EI_{eff}}{n_h}} \text{ (sands)}$$

$$D = 1.4\sqrt[4]{\frac{EI_{eff}}{0.465S_u}} \text{ (clays)}$$

n_h = soil modulus

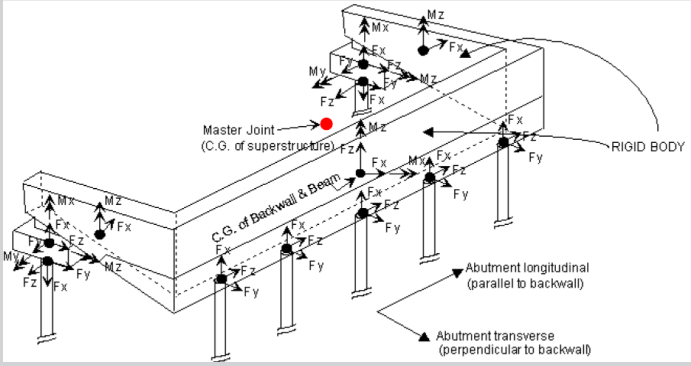
S_u = undrained shear strength

- Only for preliminary design purpose
  - Assume a single uniform soil
  - Soil is assumed perfect elastic
  - Cannot accurately assess lateral disp.
  - Don't distinguish free-headed and fixed-headed pile

Ref: AASHTO *LRFD*, Article C10.7.3.13.4 after Davisson and Robinson (1965)

Modeling stiffness matrix for abutment

- Substructuring approach
- Rigid body transformation



Learning Outcomes Review

- Modeling of pre-stressing
- Modeling of concrete creep & shrinkage
- Modeling of influence surface (discussed in Lesson 4)
- Modeling cross-frame
- Shear deformation consideration
- Overview of soil-foundation interaction

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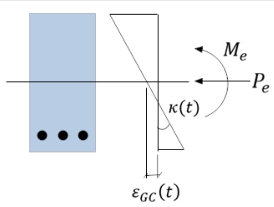
Next:
Lesson 4: Load Applications

Appendix A: Conventional structural analysis for creep & shrinkage consideration

- STEP 1: Perform cross-sectional moment-curvature ($M - \kappa$) analysis first
  - $P_e = P_i = \int_A \ \sigma(t) dA$ (equilibrium)
  - $M_e = M_i = \int_A \ y\sigma(t) dA$ (equilibrium)
  - $\varepsilon(\tau_j) = \varepsilon_s(\tau_j)$ (compatibility)
  - Stress-strain relationship:
  - $\varepsilon(\tau_j) = \varepsilon_e(\tau_j) + \varepsilon_{cr}(\tau_j) + \varepsilon_{sh}(\tau_j)$

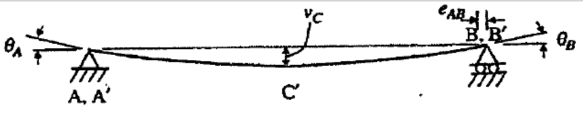
$$= \frac{1+\phi(\tau_j,\tau_0)}{E_c(\tau_0)} \sigma_c(\tau_0) + \sum_{i=1}^{j} \frac{\Delta\sigma_c(\tau_i)}{E_c(\tau_i)} [1 + \phi(\tau_j, \tau_i)] + \varepsilon_{sh}(\tau_j)$$ ✓

  - $\varepsilon(\tau_j) = \frac{\sigma_c(\tau_j)}{E_c(\tau_0)}$ for the conventional M- κ analysis ✗



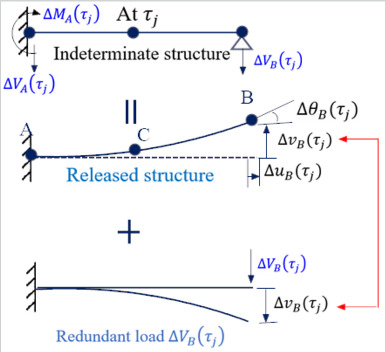
Conventional structural analysis for creep & shrinkage consideration

- STEP 2: Member nodal displacement/force analysis based on $(M-\kappa)$ analyses in Step 1
  - For statically determinate structure
  - Conjugate beam at each time step τ_j
    - $e=\int \varepsilon(x)_{\tau_j}dx$ (axial deformation)
    - $\theta=\int \kappa(x)_{\tau_j}dx$
    - $v=\iint \kappa(x)_{\tau_j}xdx$
  - For example, performing $M-\kappa$ analyses on sections $A, B, and\ C$
  - $e_{AB}=\frac{l}{6}(\varepsilon_A+4\varepsilon_C+\varepsilon_B)$
  - $\theta_A=\frac{l}{6}(\kappa_A+2\kappa_C)=-\theta_B$
  - $v_C=\frac{l^2}{96}(\kappa_A+10\kappa_C+\kappa_B)$



Conventional structural analysis for creep & shrinkage consideration

- STEP 2: Member nodal displacement/force analysis based on $(M-\kappa)$ analyses in Step 1
  - For statically indeterminate structure
  - By Conventional redundant released method
  - Conjugate beam at each time step τ_j for the released structure
  - At each time step τ_j, find incremental equivalent nodal forces $\Delta Ms, \Delta Vs, \Delta Ps$



FEM analysis for creep & shrinkage consideration

- Similar to FEM stiffness matrix formulation, the incremental equivalent nodal forces, $\{\Delta P\}$, due to creep & shrinkage for incremental time step t_{i+1} can be obtained from the concept of Principle of virtual work, $\delta U = \delta W$ as follows:
  - $\delta U = \int_0^v \sigma(\delta\epsilon)\, dv = \int_0^v \sigma([B]^T\{\delta u\})dv$; $\delta W = P\{\delta u\} = \{0\}$ (since no external load)

    $\therefore \int [B]^T \sigma dv = 0$
  - At any time increment, $\int [B]^T d\sigma dv = 0$ (1)
    - Total strain at time t: $\varepsilon = \varepsilon_e + \varepsilon_e \phi(t,\tau) + \varepsilon_s(t,\tau)$
    - $\therefore \mathrm{d}\varepsilon = d\varepsilon_e + \varepsilon_e d\phi(t,\tau) + d\varepsilon_s(t,\tau)$ or simplify to $\mathrm{d}\varepsilon = d\varepsilon_e + \varepsilon_e d\phi + d\varepsilon_s$
    - $d\varepsilon_e = \mathrm{d}\varepsilon - \varepsilon_e d\phi - d\varepsilon_s$
  - $d\sigma = Ed\varepsilon_e = E[\mathrm{d}\varepsilon - \varepsilon_e d\phi - d\varepsilon_s]$ (2)

FEM analysis for creep & shrinkage consideration

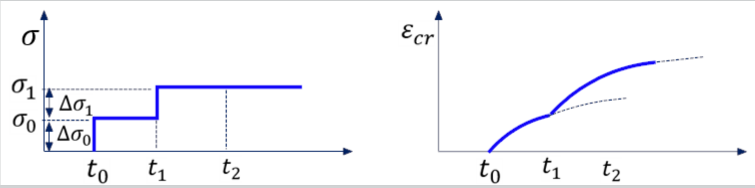
- Substitute (2) into (1):
  - $\int B^T E[\mathrm{d}\varepsilon - \varepsilon_e d\phi - d\varepsilon_s]\, dv = 0$
  - $\rightarrow \int B^T E d\varepsilon dv - \int B^T E \varepsilon_e d\phi dv - \int B^T E d\varepsilon_s dv = 0$
  - $\rightarrow \int B^T EBdudv - \int B^T \sigma d\phi dv - \int B^T E d\varepsilon_s dv = 0$
  - $\rightarrow Kdu - \int B^T \sigma d\phi dv - \int B^T E d\varepsilon_s dv = 0$
  - $\therefore Kdu = \underbrace{\int B^T \sigma d\phi dv}_{\text{Contribution from creep}} + \underbrace{\int B^T E d\varepsilon_s dv}_{\text{Contribution from shrinkage}} = \Delta P$ (equivalent nodal load) (3)
- At any time increment $i+1$, incremental nodal displacement du_{i+1} can be obtained by $du_{i+1} = K^{-1}\Delta P_i$

FEM analysis for creep & shrinkage consideration

- In which:
  - $\{\sigma\}d\phi = \sum_{j=0}^{i}\{\Delta\sigma_j\}[\phi(t_{i+1}, t_j) - \phi(t_i, t_j)]$
  - $d\varepsilon_{sh} = \varepsilon_{sh}(t_{i+1}, \tau_d) - \varepsilon_{sh}(t_i, \tau_d)$
  - t_i = time at incremental step i; t_j = time when jth load applied
  - τ_d =time when concrete starts to set
- Once du is calculated:
  - $d\varepsilon_e = Bdu$; $\mathrm{d}\varepsilon = d\varepsilon_e + \varepsilon_e d\phi + d\varepsilon_s$; $d\sigma = Ed\varepsilon_e$
  - $\varepsilon = \varepsilon + d\varepsilon$
  - $\sigma = \sigma + d\sigma$
  - Go to the next incremental time

FEM analysis for creep & shrinkage consideration

- Graphic representative of $\{\sigma\}d\phi = \sum_{j=0}^{i}\{\Delta\sigma_j\}[\phi(t_{i+1}, t_j) - \phi(t_i, t_j)]$



- $\sigma_1 d\phi = \Delta\sigma_0[\phi(t_2 - t_0) - \phi(t_1 - t_0)]$
- $+\Delta\sigma_1[\phi(t_2 - t_1) - \phi(t_1 - t_1)]$

General FEM analysis for creep & shrinkage consideration

- Advantage of FEA for creep & shrinkage:
  - On the stress level, so no cross-sectional M-Curvature analyses are needed at each time step.
  - The incremental equivalent nodal forces, $\{\Delta P\}$, can be directly obtained from Eq. (3) by numerical integration at Gauss points

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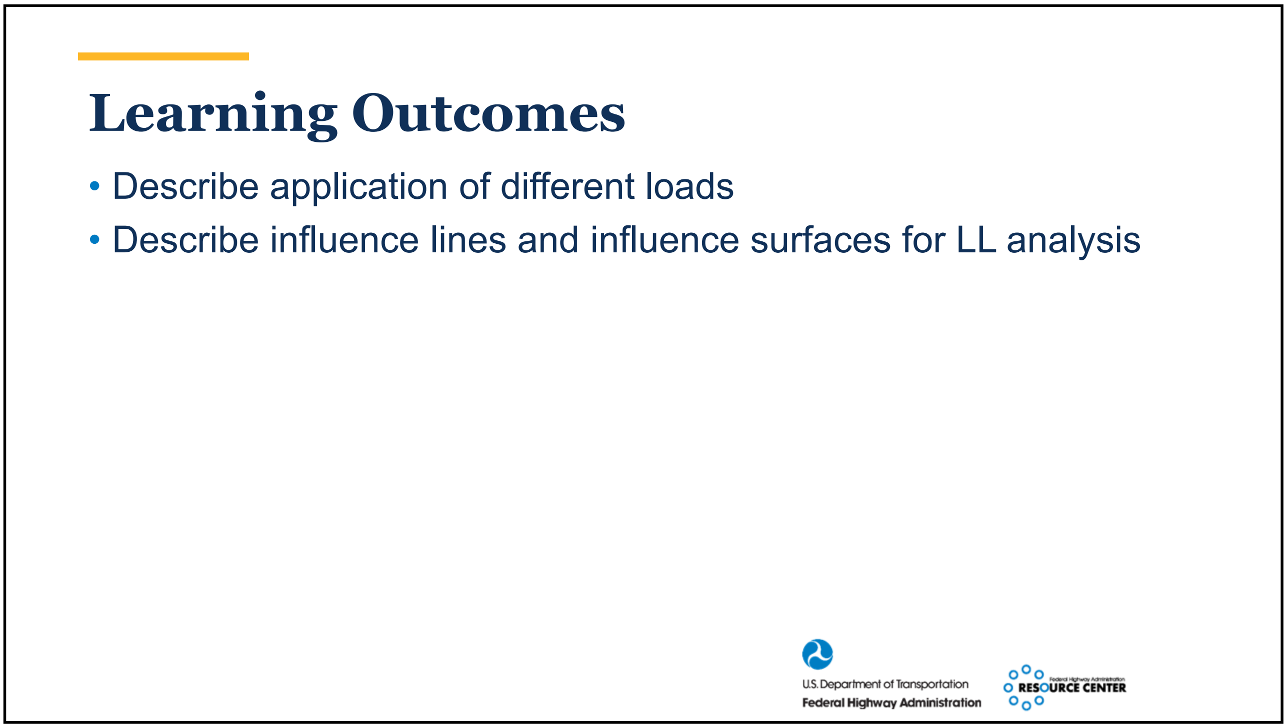
Lesson 4
Load Applications



Learning Outcomes

- Describe application of different loads
- Describe influence lines and influence surfaces for LL analysis

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Types of Loads – most software can apply:

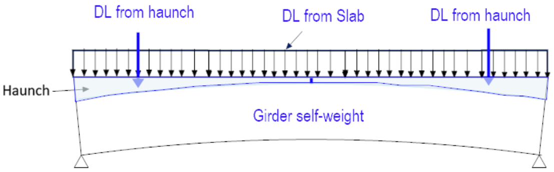
- Dead Loads
- Live Loads
- Pre-stressing Loads (Lesson 3)
- Loads due to concrete creep & shrinkage (Lesson 3)
- Loads due to temperature changes

Dead Loads

- Modeled Components – Software generates loads since cross section and properties of a member are provided.
- Non-modeled Components – DLs that designer does not want to contribute to the stiffness of a member
  - Barriers, haunches, utilities, connection loads
  - Increase material density, or
  - Through use of distributed and concentrated loads

Dead Loads

- For example: Non-composite DL1:
- Modeled component - Girder self-weight
- Non-modeled components - Slab & haunch loads

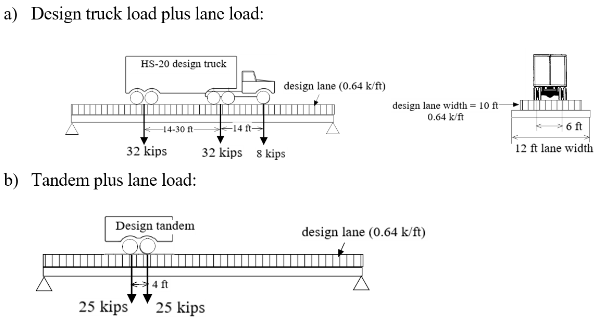


Recommendation for applying DL

- Preferably, a node should be present at locations of unmoving concentrated loads.
- The mesh should be fine enough to approximate the static equivalent load effects for distributed loads
- The finer the mesh, the more accurate the load effect is.

Live Load Models (AASHTO 3.6.1.2)

- AASHTO HL-93 Design Live Load Model: HL-93 is a notional design live load model, including:
  - HL-93 design truck;
  - HL-93 design tandem; and
  - HL-93 design lane load (uniform load)
- Actual highway lane width = $12\ ft$
- Actual design lane width = $10\ ft$ (AASHTO 3.6.1.2.4)

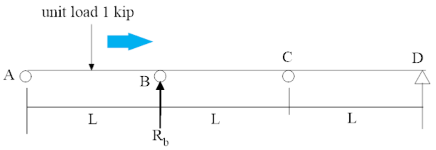


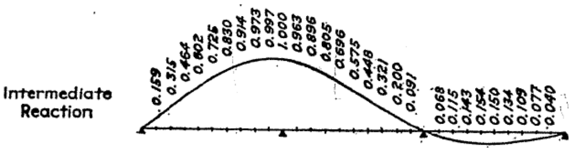
Modeling Live Loads

- Influence line
- Influence surface

Definition of influence line

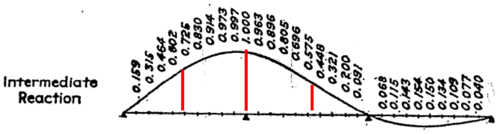
- Influence line is a 1-D mathematic function, which represents the load effect (such as moment, shear or deflection) at a specific point in a 1-D structure (a beam or a stripe of deck), due to a unit concentrate live load moving along the structure.
- For example: The reaction Influence line for a 3 equal-span continuous beam:

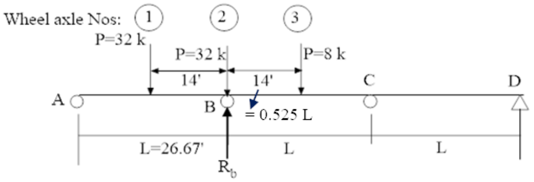




Influence Line Example:

- Use the following influence line, find the total reaction force at support B.





For wheel axle 3: $\left[0.448 + (0.575 - 0.448)\left(\frac{3}{4}\right)\right](8)$ = 4.346 (kip)

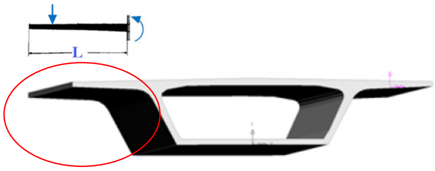
For wheel axle 2: (1.0)(32)__________________________= ___32_________

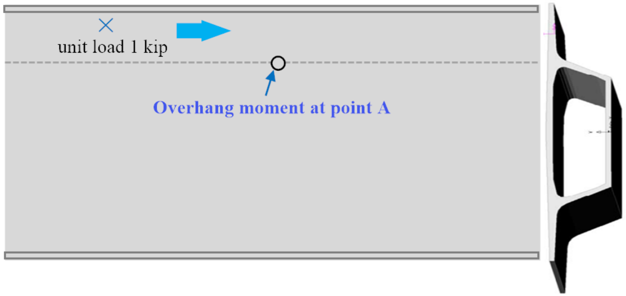
For wheel axle 1: $\left|0.602 + (0.725 - 0.602)\left(\frac{3}{4}\right)\right|(32)$____= _22.126_______

Total = _58.56 (kip)_____

Definition of influence surface

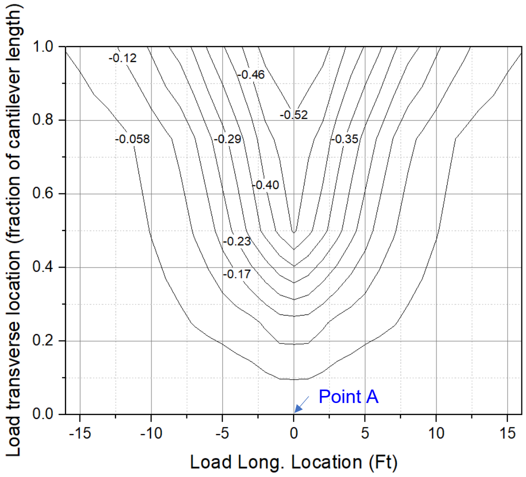
- Influence surface is a 2-D mathematic function, which represents the load effect (such as moment, shear or deflection) at a specific point in a 2-D structure (such as a deck surface), due to a unit concentrate live load moving on the surface.





Definition of influence surface

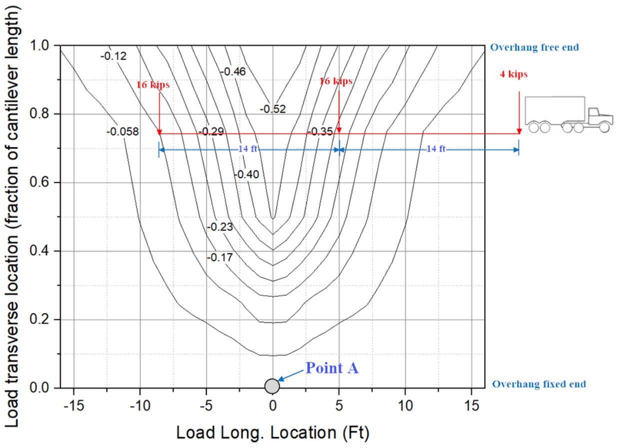
- For example: bending moment influence surface at Point A for deck overhang of a segmental box superstructure.



Influence Surface Example:

- Use the following influence line, find the total reaction force at Point A.
- Solution:

$$\begin{aligned} M_{point\ A} &= 16(0.12) + 16(0.27) \\ &+4(0) = 6.24\ (k - ft) \end{aligned}$$



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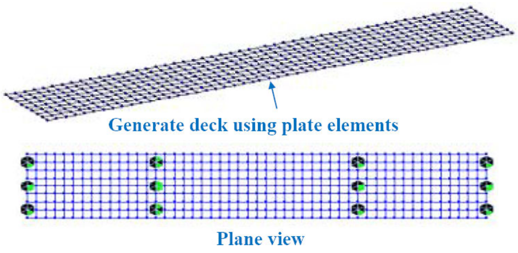
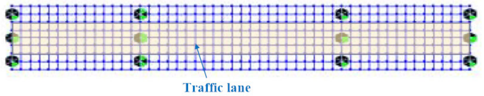
How to generate an influence surface?

- Step 1: Model the structural deck by plate elements


Generate deck using plate elements

Plane view

- Step 2: Define the traffic lane(s) location


Traffic lane

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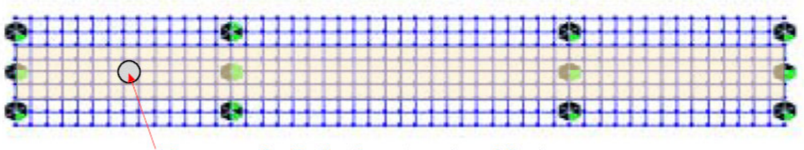
How to generate an influence surface?

- Step 3: Define force (or displacement) type at a specific point (or location) for the influence surface

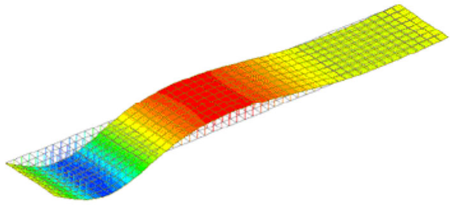


For example: Deflection at center of first span

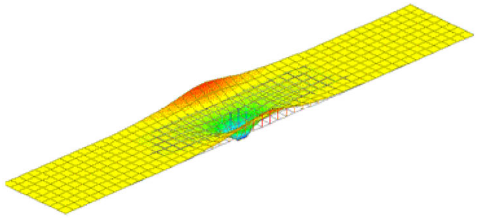
- Step 4: Generate influence surface - by applying the unit load at each note (one at a time) within the traffic lane, and save the force/displacement at the point considered in the data base.

How to generate an influence surface?

- Step 4: Generate influence surfaces (continued)

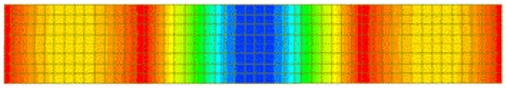


Influence surface of the displacement at the center of the 1st span

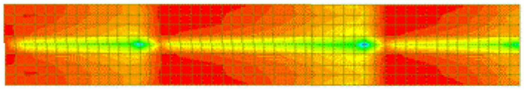


Influence surface for moment (M_x) at the center of the 2nd span

Use influence surface for live load analysis



Maximum displacement caused by HL-93 truck load



Maximum shear caused by HL-93 truck load

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Verifying Live Loads

- The designer is responsible to confirm results
- Spot check results or use separate software tool to confirm results

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Loads due to temperature changes

- The incremental equivalent nodal forces, $\{\Delta P\}$, due to temperature change, ΔT.
- Principle of virtual work
- $\{\Delta P\} = \int [B]^T [E] \{d\varepsilon_t\} dv$

Contribution from
Strain change due to temperature

  - where,
    - $\{d\varepsilon_t\} = \{\alpha \Delta T\}$
    - $\alpha =$ Thermal coefficient;

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Loads due to temperature changes

- or at Gauss points (Use 2D element as an example):
  - $\{\Delta P\} = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} W_i W_j [B]_{ij}^{T} [E] \{d\varepsilon_t\} |J_{ij}|$;

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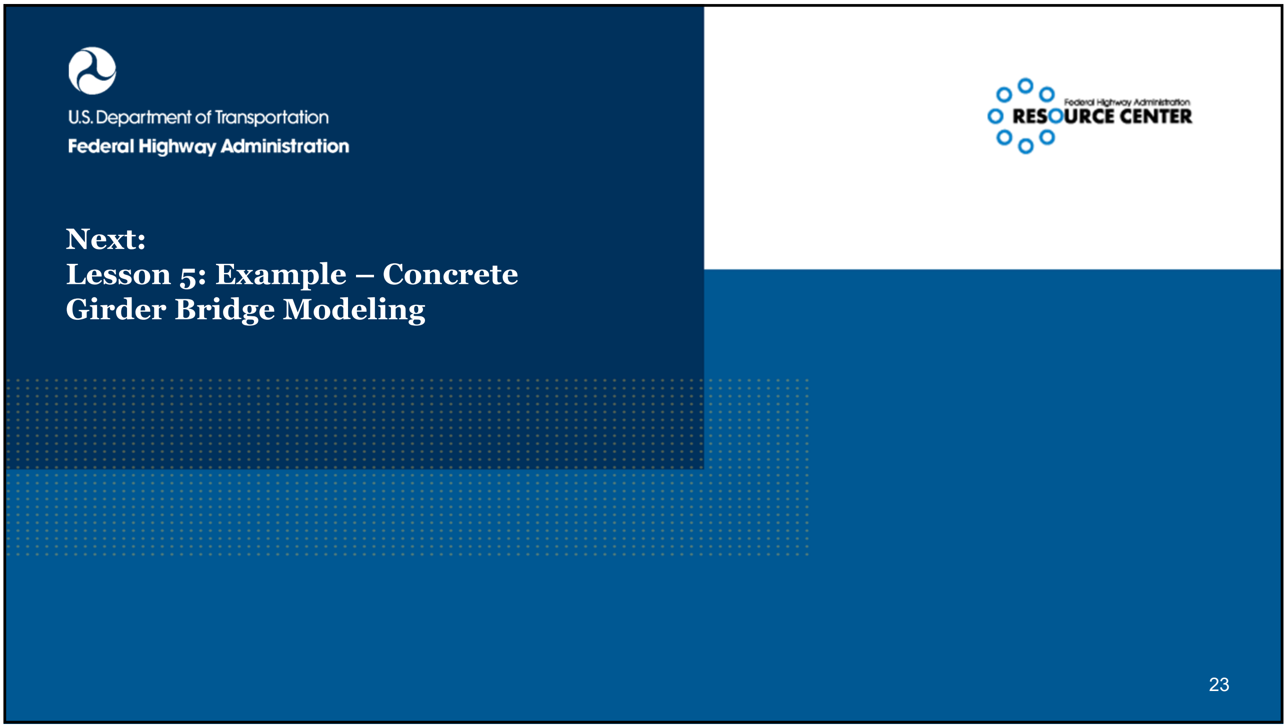
- **Software demonstration of influence surfaces and vehicular loads optimization**

Learning Outcomes

- Describe application of different loads
  - Dead Loads
  - Live Loads
  - Pre-stressing Loads (Lesson 3)
  - Loads due to concrete creep & shrinkage (Lesson 3)
  - Loads due to temperature changes
- Describe influence lines and influence surfaces for LL analysis
- Demonstrate how typical commercial bridge software handles influence surfaces and moving loads analysis



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Next:
Lesson 5: Example – Concrete Girder Bridge Modeling

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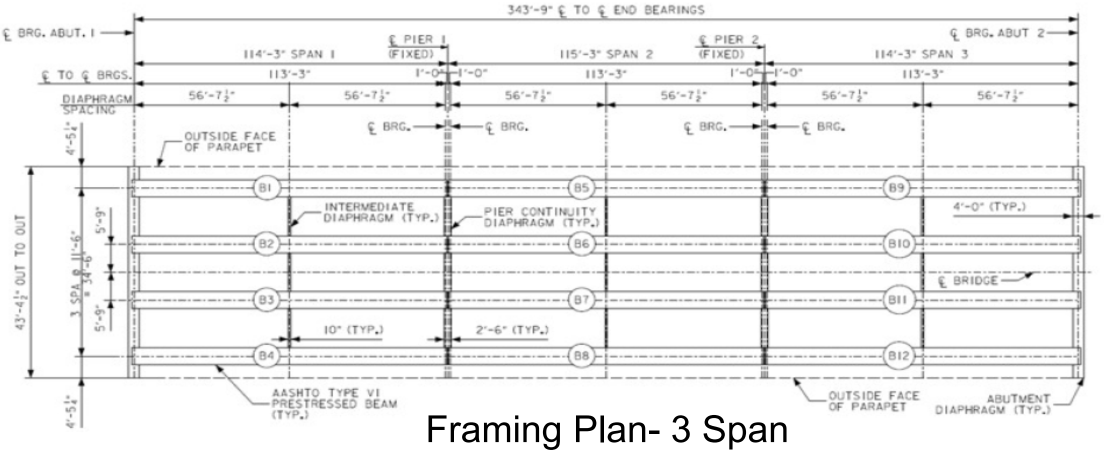
Lesson 5
Concrete PS Girder Bridges Modeling

Learning Outcome

By the end of this lesson, you will be able to:

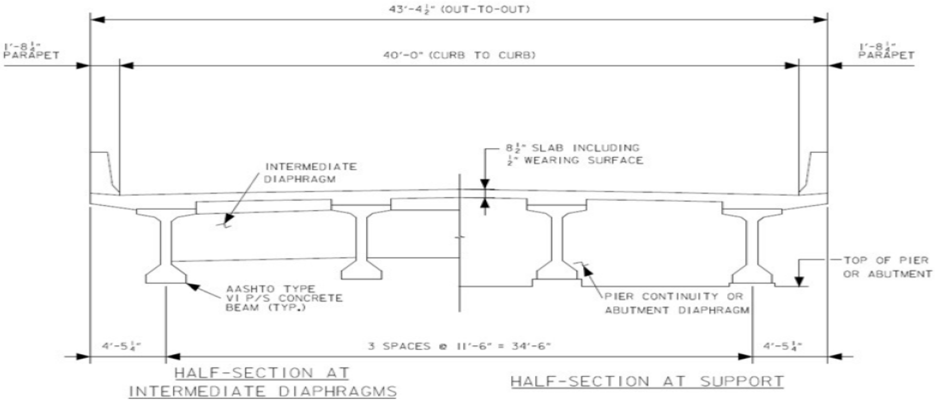
1. Perform 1D line Girder Analysis of three span bridge
2. Perform 2D Plate and Eccentric Beam (PEB) Analysis of three span bridge
3. Perform 3D Finite Element Analysis of 3 span bridge
4. Compare results from the three model analysis

Precast Concrete I- Girder Bridge

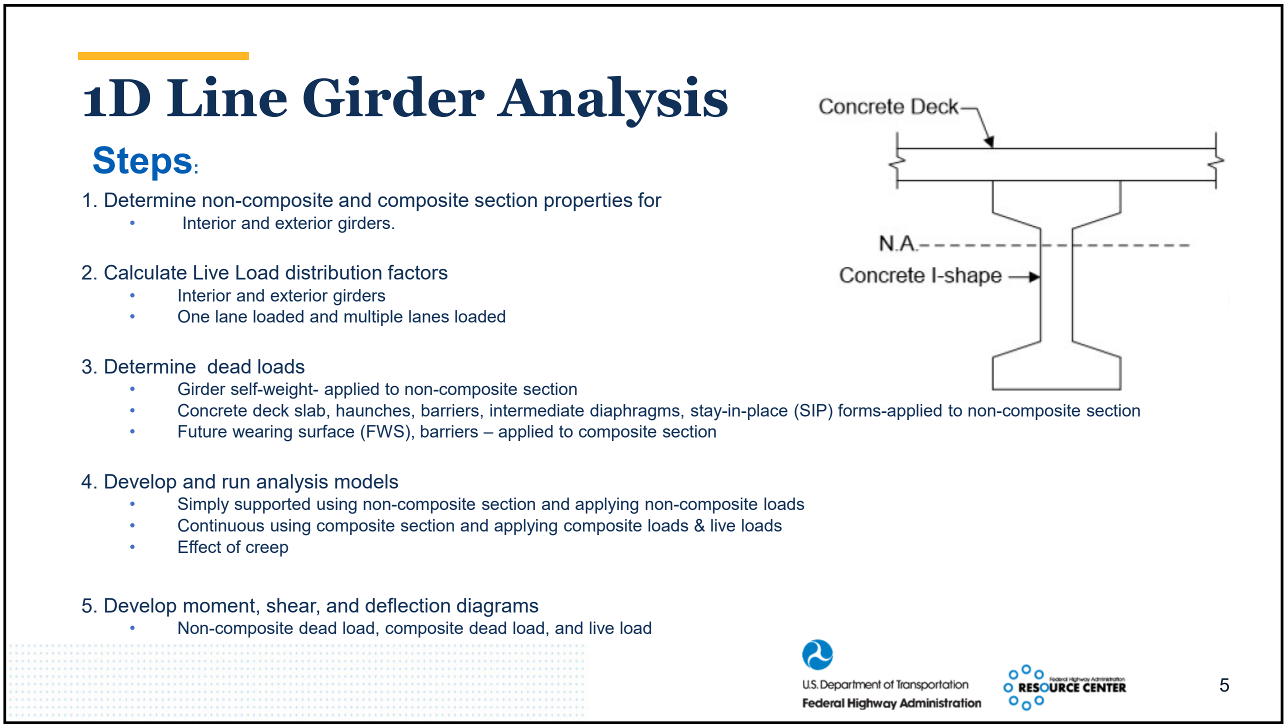


Framing Plan- 3 Span

Precast Concrete I- Girder Bridge



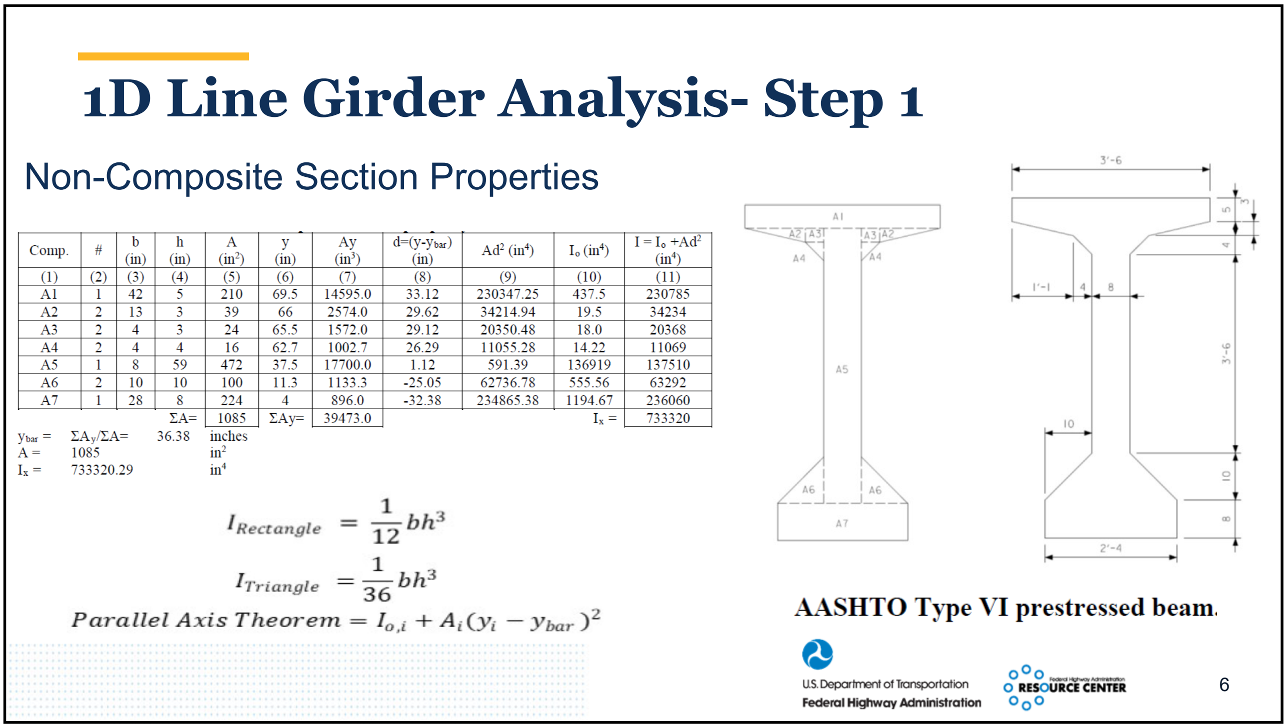
Cross-Section



1D Line Girder Analysis

Steps:

1. Determine non-composite and composite section properties for
    - Interior and exterior girders.
2. Calculate Live Load distribution factors
    - Interior and exterior girders
    - One lane loaded and multiple lanes loaded
3. Determine dead loads
    - Girder self-weight- applied to non-composite section
    - Concrete deck slab, haunches, barriers, intermediate diaphragms, stay-in-place (SIP) forms-applied to non-composite section
    - Future wearing surface (FWS), barriers – applied to composite section
4. Develop and run analysis models
    - Simply supported using non-composite section and applying non-composite loads
    - Continuous using composite section and applying composite loads & live loads
    - Effect of creep
5. Develop moment, shear, and deflection diagrams
    - Non-composite dead load, composite dead load, and live load



1D Line Girder Analysis- Step 1

Non-Composite Section Properties

Comp.	#	b (in)	h (in)	A (in^2)	y (in)	Ay (in^3)	d=(y-y_{bar}) (in)	Ad^2 (in^4)	I_o (in^4)	I = I_o +Ad^2 (in^4)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
A1	1	42	5	210	69.5	14595.0	33.12	230347.25	437.5	230785
A2	2	13	3	39	66	2574.0	29.62	34214.94	19.5	34234
A3	2	4	3	24	65.5	1572.0	29.12	20350.48	18.0	20368
A4	2	4	4	16	62.7	1002.7	26.29	11055.28	14.22	11069
A5	1	8	59	472	37.5	17700.0	1.12	591.39	136919	137510
A6	2	10	10	100	11.3	1133.3	-25.05	62736.78	555.56	63292
A7	1	28	8	224	4	896.0	-32.38	234865.38	1194.67	236060
			ΣA=	1085	ΣAy=	39473.0			I_x =	733320

y_{bar} = $\Sigma A_y/\Sigma A$= 36.38 inches
A = 1085 in^2
I_x = 733320.29 in^4

$$I_{Rectangle} = \frac{1}{12}bh^3$$

$$I_{Triangle} = \frac{1}{36}bh^3$$

$$Parallel\ Axis\ Theorem = I_{o,i} + A_i(y_i - y_{bar})^2$$

AASHTO Type VI prestressed beam.

1D Line Girder Analysis- Step 1

Composite Section Properties

Comp.	b (in)	h (in)	A (in^2)	y (in)	Ay (in^3)	d=(y-y$_{bar}$) (in)	Ad^2 (in^4)	I$_o$ (in^4)	I = I$_o$ +Ad^2 (in^4)
(1)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Deck	94.52	8	756.2	76	57468.9	23.35	412198.0	4032.90	416230.9
Beam	-	-	1085	36.38	39473	-16.27	287273.2	733320.3	1020593.5
		ΣA=	1841.2	ΣAy=	96941.9			I$_x$ =	1436824.3

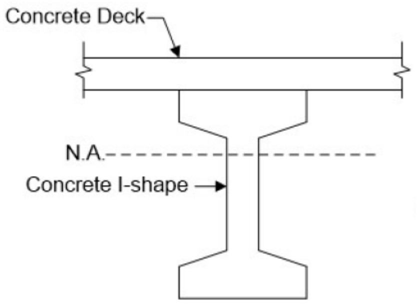
y_{bar} = $\Sigma A_y/\Sigma A$= 52.65 inches
A = 1841.2 in^2
I_x = 1436824.3 in^4

interior girder.

Comp.	b (in)	h (in)	A (in^2)	y (in)	Ay (in^3)	d=(y-y$_{bar}$) (in)	Ad^2 (in^4)	I$_o$ (in^4)	I = I$_o$ +Ad^2 (in^4)
(1)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Deck	83.73	8	669.9	76	50909.9	24.50	401952.3	3572.6	405524.9
Beam	-	-	1085	36.38	39473	-15.12	248161.0	733320.3	981481.3
		ΣA=	1754.9	ΣAy=	90382.9			I$_x$ =	1387006.2

y_{bar} = $\Sigma A_y/\Sigma A$= 51.50 inches
A = 1754.9 in^2
I_x = 1387006.2 in^4

exterior girder.



$$E_c = 1820\sqrt{f'_c} \qquad w_c = 0.140 + 0.001 f'_c$$

$$E_c = 33{,}000 K_1 w_c^{1.5}\sqrt{f'_c} \qquad n = \frac{E_{beam}}{E_{deck}}$$

1D Line Girder Analysis- Step 2

Live Load Distribution Factors – interior girders

$$g_{SIM} = 0.06 + \left(\frac{S}{14}\right)^{0.4}\left(\frac{S}{L}\right)^{0.3}\left(\frac{K_g}{12.0Lt_s^3}\right)^{0.1}$$

$$g_{MIM} = 0.075 + \left(\frac{S}{9.5}\right)^{0.6}\left(\frac{S}{L}\right)^{0.2}\left(\frac{K_g}{12.0Lt_s^3}\right)^{0.1}$$

Action	Lanes Loaded	Spans 1 and 3	Span 2
M+ and M- not between POCs	1	0.61	0.60
	2+	0.91	0.91
M- between POCs	1	0.61	
	2+	0.91	

Where: S = girder spacing [ft]
L = span length [ft]
t_s = deck slab thickness [in]
K_g = longitudinal stiffness parameter [in^4] = n(I_x + Ae_g^2)
n = modular ratio
A, I_x = area and moment of inertia for non-composite beam
e_g = distance between centers of gravity of the non-composite beam and deck slab [in]
g = distribution factor

1D Line Girder Analysis- Step 2

Live Load Distribution Factors – Exterior girders

Lever rule method

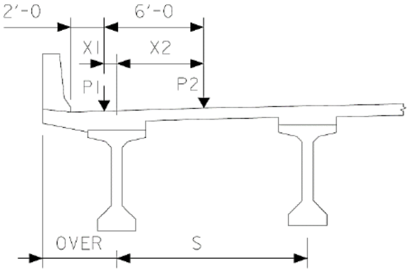
$$X1 = OVER - Barrier - 2'$$

$$X2 = 6' + X1$$

$$g = \frac{m(S - X1 + S - X2)}{2S}$$

$$g_{MEM} = e g_{MIM}$$

$$e = 0.77 + \frac{d_e}{9.1}$$



Lever rule dimensions.

Action	Lanes Loaded	Spans 1 and 3	Span 2
M+ and M- not between POCs	2+	0.97	0.97
M- between POCs	2+	0.97	

1D Line Girder Analysis- Step2

Live Load Distribution Factors – Exterior girders

Rigid cross- section method

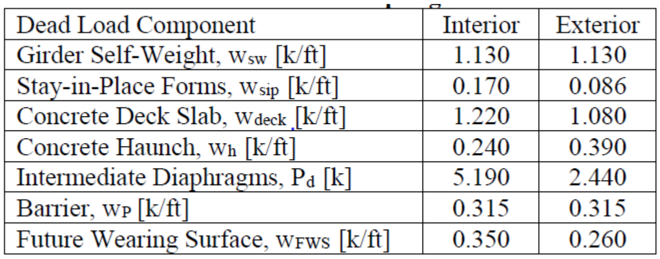
$$R = m\left(\frac{N_L}{N_b} + \frac{X_{ext}\sum_1^{N_L} e}{\sum_1^{N_b} x^2}\right)$$

Where:
R = reaction on exterior beam in terms of lanes
m = multiple presence factor, from AASHTO LRFD Table 3.6.1.1.2-1
N_L = number of loaded lanes under consideration
e = eccentricity of design truck or design lane load from center of gravity of the pattern of girders [ft]
x = horizontal distance from center of gravity of girder pattern to each girder [ft]
X_{ext} = horizontal distance from center of gravity of girder pattern to exterior girder [ft]
N_b = number of beams/girders

	One Lane Loaded	Multiple Lanes Loaded
Table 4.6.2.2.2d-1	0.97	0.97
Rigid Cross-Section	0.77	0.97

1D Line Girder Analysis- Step3

Dead loads



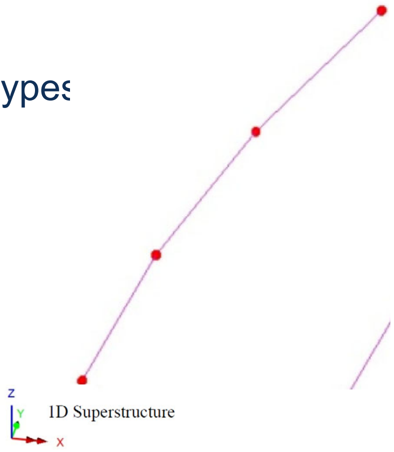
Dead Load Component	Interior	Exterior
Girder Self-Weight, w_{sw} [k/ft]	1.130	1.130
Stay-in-Place Forms, w_{sip} [k/ft]	0.170	0.086
Concrete Deck Slab, w_{deck} [k/ft]	1.220	1.080
Concrete Haunch, w_h [k/ft]	0.240	0.390
Intermediate Diaphragms, P_d [k]	5.190	2.440
Barrier, w_P [k/ft]	0.315	0.315
Future Wearing Surface, w_{FWS} [k/ft]	0.350	0.260

Dead loads on a per girder basis.

1D Line Girder Analysis-Step4

Develop Analysis Models

1. Basic Layout – Span lengths and support types
2. Properties of cross-sections
3. Dead loads
4. Live loads
5. Verify model is correct- Simple check



1D Superstructure

Question # 1 (L05S13Q1)

What is a simple check to verify 1D model. Select all that apply

A. Compare DL Moment to AISC tables: Moment, Shears, and Reactions for Continuous Highway Bridges
B. Compare simple span DL Moments, Shears and reactions to AISC Steel Construction Manual tables
C. Apply dead load of barrier and compare to reactions from 1 D Analysis
D. All of the above

Question # 1

Answer: All of the above

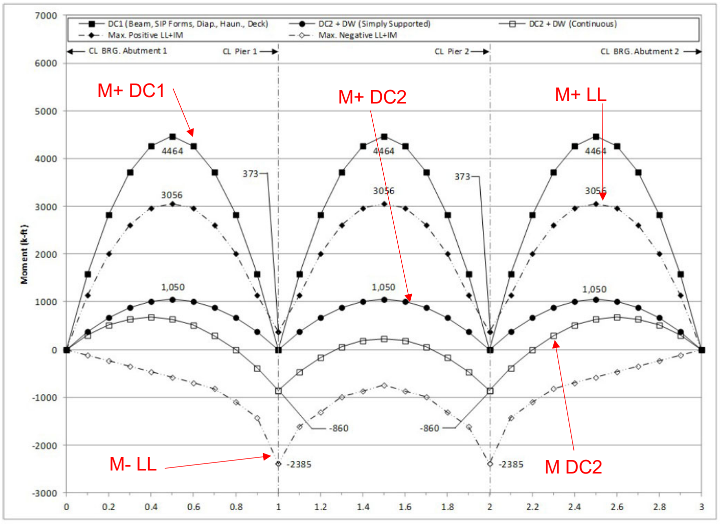
Example: C

Total Applied Load = 0.315 k/ft × 343.75 ft = 108.28 k

Abutment 1	Pier 1	Pier 2	Abutment 2	Total
14.38 k	39.76 k	39.76 k	14.38 k	108.28 k

1D Line Girder Analysis- Step5

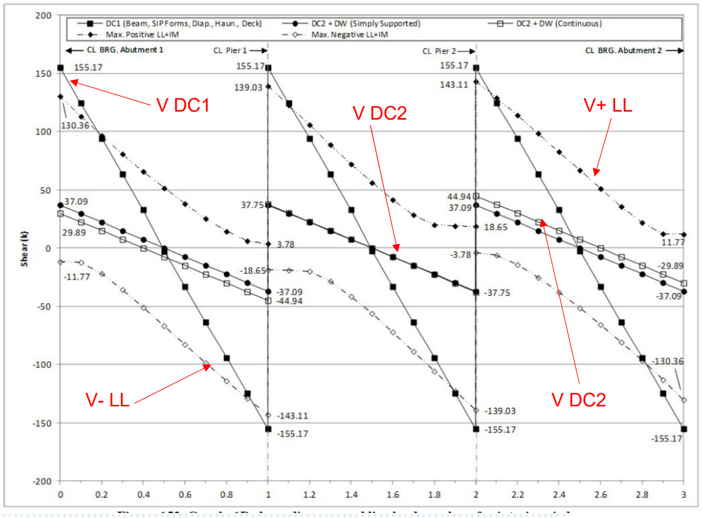
Develop moment, shear and deflection diagrams



Moment diagram and live load envelop for interior girder

1D Line Girder Analysis- Step5

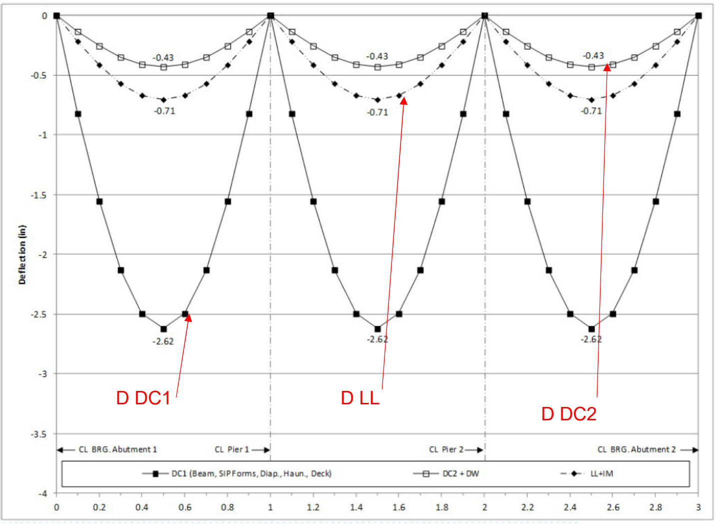
Develop moment, shear and deflection diagrams



Shear diagram and live load envelop for interior girder

1D Line Girder Analysis- Step5

Develop moment, shear and deflection diagrams



Deflection diagram and live load envelop for interior girder

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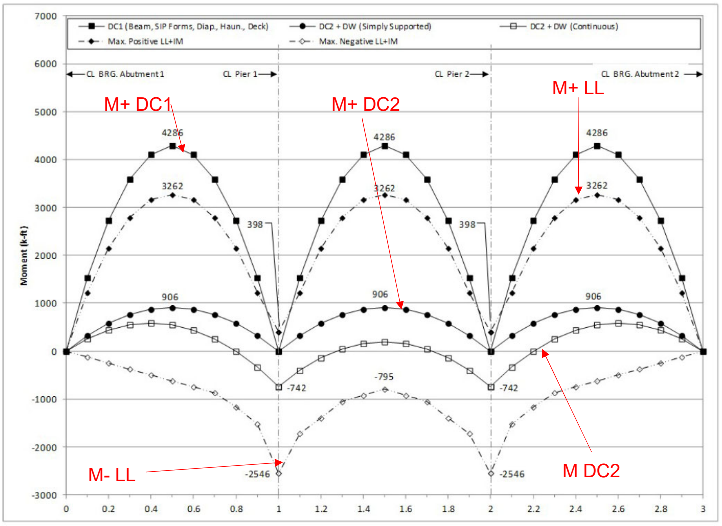
1D Line Girder Analysis- Step5

Develop moment, shear and deflection diagrams



Moment diagram and live load envelop for exterior girder

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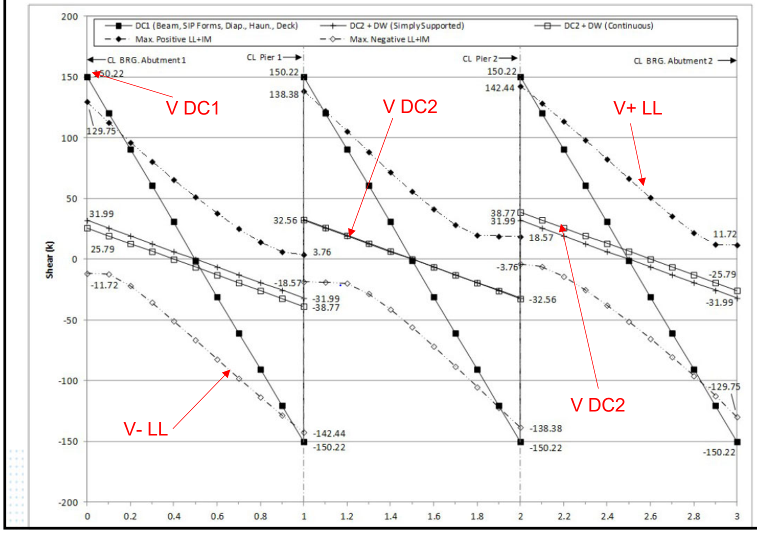
1D Line Girder Analysis- Step5

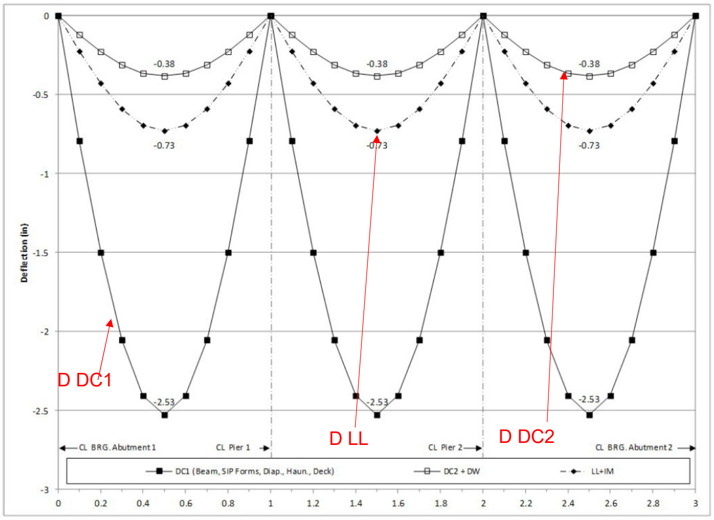
Develop moment, shear and deflection diagrams



Shear diagram and live load envelop for exterior girder

1D Line Girder Analysis- Step5

Develop moment, shear and deflection diagrams

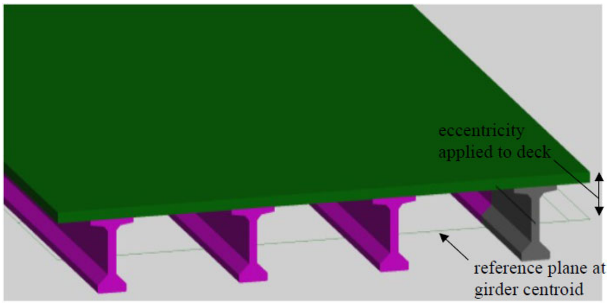


Deflection diagram for exterior girder

2D Plate and Eccentric Beam Analysis

Steps:

1. Create a Model for non-composite dead loads
2. Create a Model for composite dead loads
3. Create a Model for live load
4. Combine analysis results



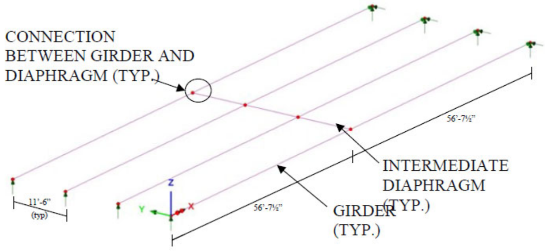
2D Plate and Eccentric Beam Analysis

Step 1: Create a model for non-composite

Step 1a- Define girder and intermediate diaphragm locations

Component	Start			End		
	x (ft)	y (ft)	z (ft)	x (ft)	y (ft)	z (ft)
B1	0	34.5	0	113.25	34.5	0
B2	0	23.0	0	113.25	23.0	0
B3	0	11.5	0	113.25	11.5	0
B4	0	0	0	113.25	0	0
Int. Diaphragm B1-B2	56.625	23.0	0	56.625	34.5	0
Int. Diaphragm B2-B3	56.625	11.5	0	56.625	23.0	0
Int. Diaphragm B3-B4	56.625	0	0	56.625	11.5	0

Coordinates for girder and intermediate diaphragm ends.



Girder and intermediate diaphragm location.

2D Plate and Eccentric Beam Analysis

Step 1b – Define Girder and Intermediate Diaphragm Cross-Sections

Section Property	Value
Cross-section Area (A) (ft^2)	3.194
Strong Axis Moment of Inertia (I_{yy}) (ft^4)	3.912
Weak Axis Moment of Inertia (I_{zz}) (ft^4)	0.185
Torsion Constant (J_{xx}) (ft^4)	0.638
Shear Area in y direction (A_{vy}) (ft^2)	2.662
Shear Area in z direction (A_{vz}) (ft^2)	2.662
Offset in z direction (R_z) (ft)	-0.635

Intermediate diaphragm section properties.

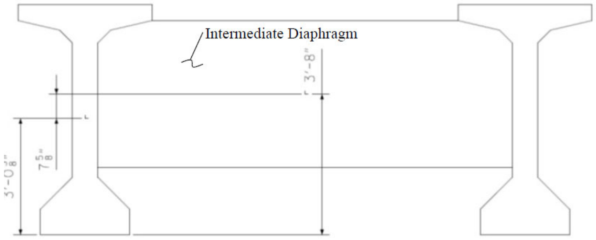


Illustration. Diaphragm eccentricity

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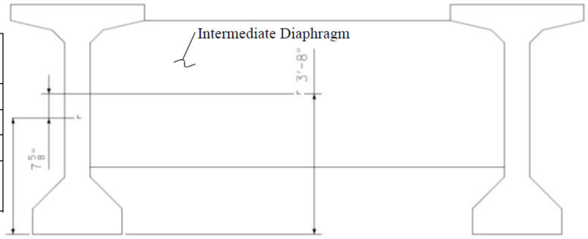
2D Plate and Eccentric Beam Analysis

Step 1c – Define Material Properties for Girders and Intermediate Diaphragms

Material Property	Girder Concrete (8 ksi)	Int. Diaphragm Concrete (3.5 ksi)
Modulus of Elasticity (ksf)	765,216	490,307
Poisson's Ratio	0.2	0.2
Unit Weight (k/ft^3)	0.153	0.150
Thermal Expansion Coefficient (ft/ft/°F)	6.0E-6	6.0E-6

Concrete material properties.



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2D Plate and Eccentric Beam Analysis

Step 1d – Define Support Conditions

- Simply supported
- One support of each girder is restrained vertically and transversely, other end is restrained vertically, transversely, and longitudinally

Step 1e – Define Non-Composite Loads

- self-weight (SIP forms, haunches, and deck slab)

Step 1f – Define Load Cases- Non-composite Load can be combined into one load case

2D Plate and Eccentric Beam Analysis

Step 1g – Ensure Correct Attributes Are Assigned to Components

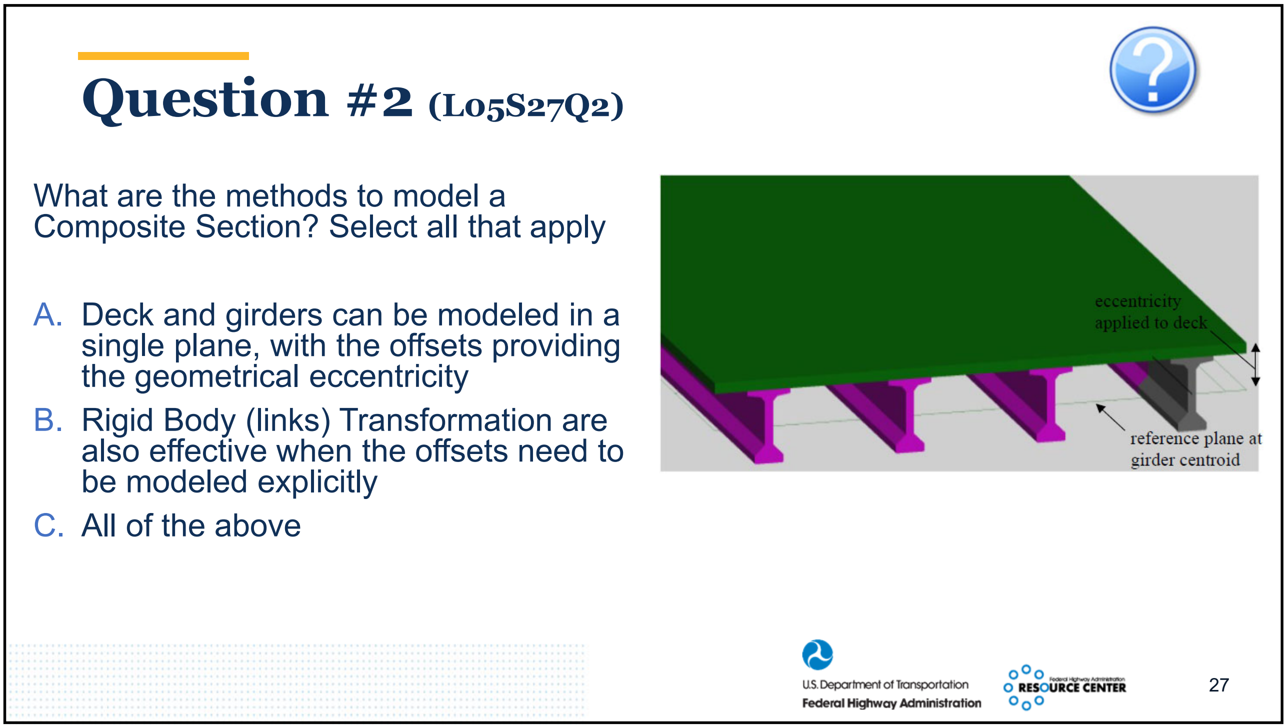
Girders

- Beam elements
- Geometric cross-section
- Concrete material properties, f’c = 8 ksi in this example
- Dead loads (including self-weight, SIP forms, haunches, and deck slab)

Intermediate diaphragms

- Beam elements
- Geometric cross-section
- Concrete material properties, f’c = 3.5 ksi in this example
- Dead load (self-weight)

Step 1h – Run Analysis and Verify Results



Question #2 (L05S27Q2)

What are the methods to model a Composite Section? Select all that apply

A. Deck and girders can be modeled in a single plane, with the offsets providing the geometrical eccentricity
B. Rigid Body (links) Transformation are also effective when the offsets need to be modeled explicitly
C. All of the above

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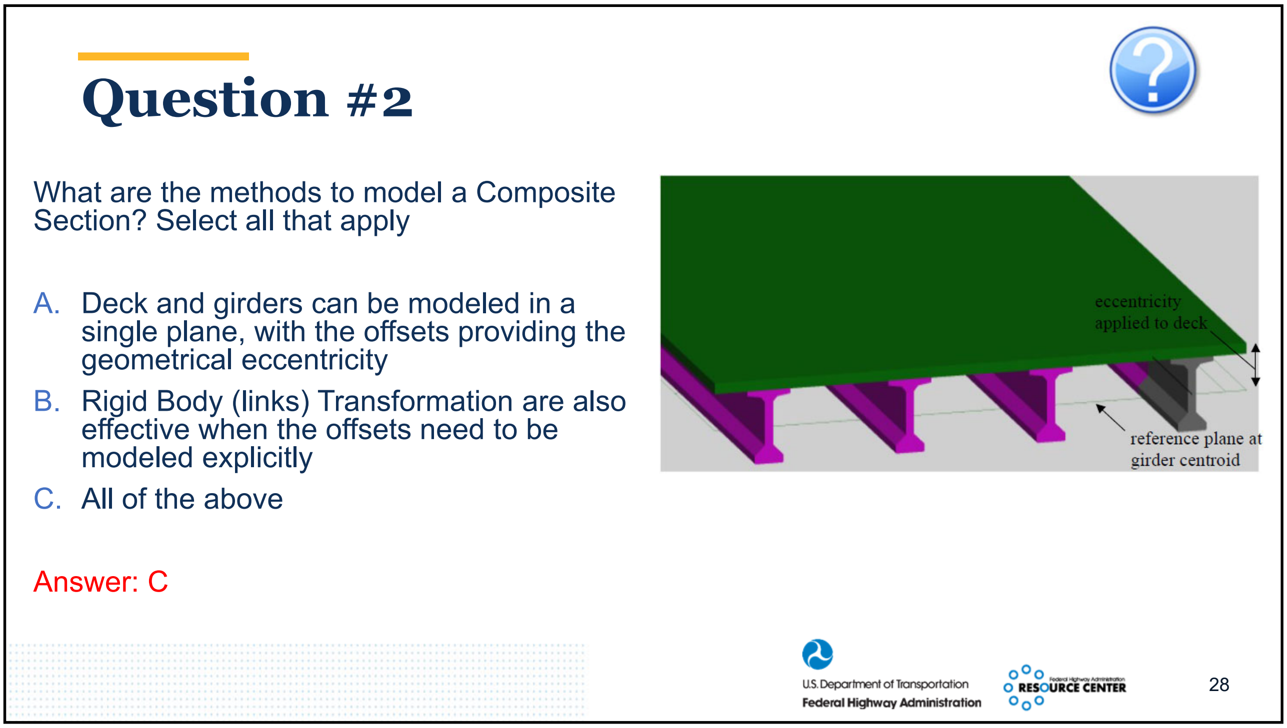

Question #2

What are the methods to model a Composite Section? Select all that apply

A. Deck and girders can be modeled in a single plane, with the offsets providing the geometrical eccentricity
B. Rigid Body (links) Transformation are also effective when the offsets need to be modeled explicitly
C. All of the above

Answer: C

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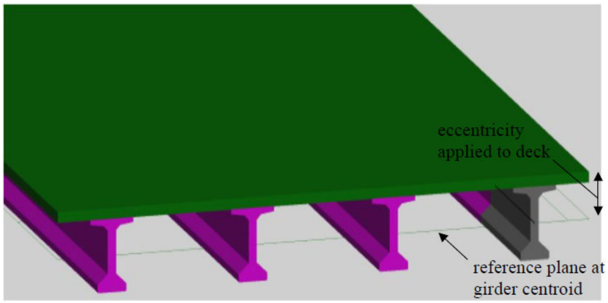
2D Plate and Eccentric Beam Analysis

Step 2 – Create Composite Dead Load Model

Step 2a – Define Girder, Diaphragm, and Concrete Deck Slab Location

Table 20. Coordinates for girder and diaphragm ends.

Component		Start			End		
		x (ft)	y (ft)	z (ft)	x (ft)	y (ft)	z (ft)
Abut. 1	Abut. 1 Diaphragm B1-B2	0	27.4375	0	0	38.9375	0
	Abut. 1 Diaphragm B2-B3	0	15.9375	0	0	27.4375	0
	Abut. 1 Diaphragm B3-B4	0	4.4375	0	0	15.9375	0
Span 1	B1	0	38.9375	0	114.25	38.9375	0
	B2	0	27.4375	0	114.25	27.4375	0
	B3	0	15.9375	0	114.25	15.9375	0
	B4	0	4.4375	0	114.25	4.4375	0
	Int. Diaphragm B1-B2	57.125	27.4375	0	57.125	38.9375	0
	Int. Diaphragm B2-B3	57.125	15.9375	0	57.125	27.4375	0
	Int. Diaphragm B3-B4	57.125	4.4375	0	57.125	15.9375	0
Pier 1	P1 Diaphragm B1-B2	114.25	27.4375	0	114.25	38.9375	0
	P1 Diaphragm B2-B3	114.25	15.9375	0	114.25	27.4375	0
	P1 Diaphragm B3-B4	114.25	4.4375	0	114.25	15.9375	0
Span 2	B5	114.25	38.9375	0	229.5	38.9375	0
	B6	114.25	27.4375	0	229.5	27.4375	0
	B7	114.25	15.9375	0	229.5	15.9375	0
	B8	114.25	4.4375	0	229.5	4.4375	0
	Int. Diaphragm B5-B6	171.875	27.4375	0	171.875	38.9375	0
	Int. Diaphragm B6-B7	171.875	15.9375	0	171.875	27.4375	0
	Int. Diaphragm B7-B8	171.875	4.4375	0	171.875	15.9375	0
Pier 2	P2 Diaphragm B1-B2	229.5	27.4375	0	229.5	38.9375	0
	P2 Diaphragm B2-B3	229.5	15.9375	0	229.5	27.4375	0
	P2 Diaphragm B3-B4	229.5	4.4375	0	229.5	15.9375	0
Span 3	B9	229.5	38.9375	0	343.75	38.9375	0
	B10	229.5	27.4375	0	343.75	27.4375	0
	B11	229.5	15.9375	0	343.75	15.9375	0
	B12	229.5	4.4375	0	343.75	4.4375	0
	Int. Diaphragm B9-B10	286.625	27.4375	0	286.625	38.9375	0
	Int. Diaphragm B10-B11	286.625	15.9375	0	286.625	27.4375	0
	Int. Diaphragm B11-B12	286.625	4.4375	0	286.625	15.9375	0
Abut. 2	Abut. 2 Diaphragm B1-B2	343.75	27.4375	0	343.75	38.9375	0
	Abut. 2 Diaphragm B2-B3	343.75	15.9375	0	343.75	27.4375	0
	Abut. 2 Diaphragm B3-B4	343.75	4.4375	0	343.75	15.9375	0



Corner	x (ft)	y (ft)	z (ft)
Upper, Left	0	43.375	0
Upper, Right	343.75	43.375	0
Lower, Right	343.75	0	0
Lower, Left	0	0	0

Coordinates of concrete deck slab corners

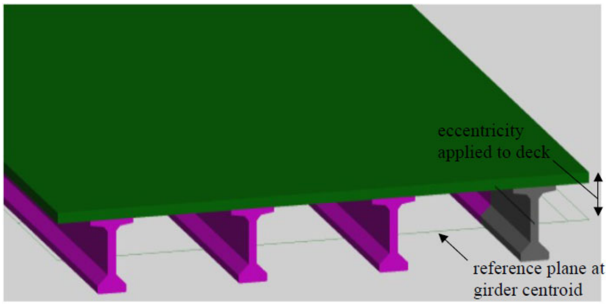
2D Plate and Eccentric Beam Analysis

Step 2b – Define Diaphragm cross- sections and concrete deck slab thickness

Deck thickness= 8”, Eccentricity= 3.406’

Section Property	Pier	Abutment
Cross-section Area (A) (ft^2)	17.98	26.00
Strong Axis Moment of Inertia (I_{yy}) (ft^4)	77.49	91.54
Weak Axis Moment of Inertia (I_{zz}) (ft^4)	9.36	34.67
Torsion Constant (J_{xx}) (ft^4)	29.26	85.55
Shear Area in y direction (A_{vy}) (ft^2)	14.98	21.67
Shear Area in z direction (A_{vz}) (ft^2)	14.98	21.67
Offset in z direction (R_z) (ft)	0.625	0.281

Pier and abutment diaphragm section properties.

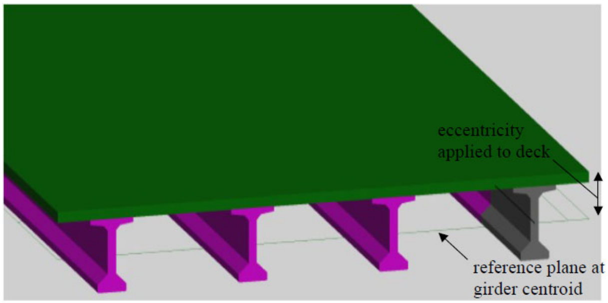


2D Plate and Eccentric Beam Analysis

Step 2c – Define Material Properties for the Concrete Deck Slab

Material Property	Deck Slab Concrete (4 ksi)
Modulus of Elasticity (ksf)	524,757
Poisson's Ratio	0.2
Unit Weight (k/ft^3)	0.159
Thermal Expansion Coefficient (ft/°F)	6.0E-6

Concrete material properties.



2D Plate and Eccentric Beam Analysis

Step 2d – Define Support Conditions

- Abutments and Pier 1 restrain vertical and transverse directions
- Pier 2, restrained vertically, transversely, and longitudinally

Step 2e – Define Dead Loads Applied to Composite Structure

- FWS and Barriers (Already defined)

Step 2f – Define Load Cases

- Separate load cases for FWS (DW) and Barriers(DC)

Step 2g – Ensure Correct Attributes Are Assigned to Components

- Girders
  - Beam Elements
  - Geometric cross-section
  - Concrete material properties, f'c =8ksi
- Intermediate Diaphragm
  - Beam Elements
  - Geometric cross-section
  - Concrete material properties, f'c =3.5ksi
- Pier Diaphragms
  - Beam Elements
  - Geometric cross-section
  - Concrete material properties, f'c =3.5 ksi
- Concrete Deck Slab
  - Thick shell elements
  - Geometric cross-section
  - Concrete material properties, f'c=4 ksi

Step 2h – Run Analysis and Verify Results using Simplified Methods

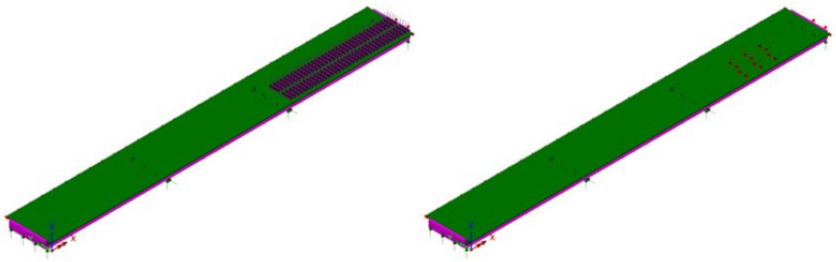
2D Plate and Eccentric Beam Analysis

Step 3 – Create Live Load Model

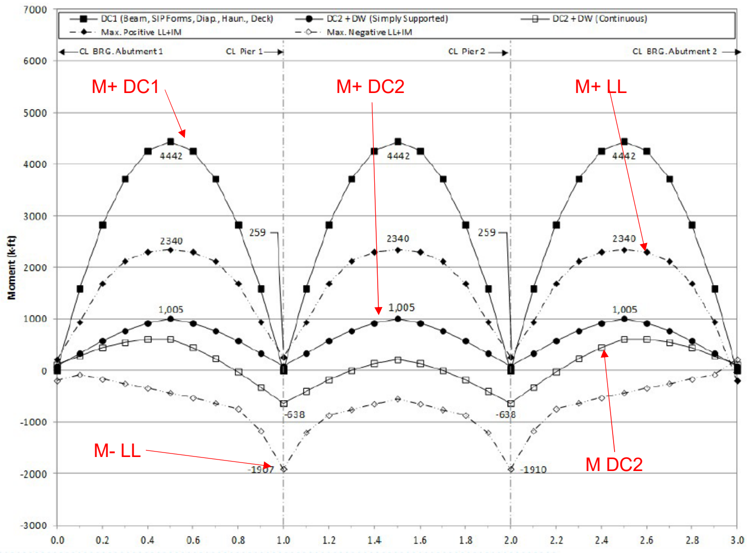
Step 3a – Determine Optimized Live Load

Step 3b – Run Analysis and Verify Results using Simplified Methods

Step 4 – Combine Analysis Results

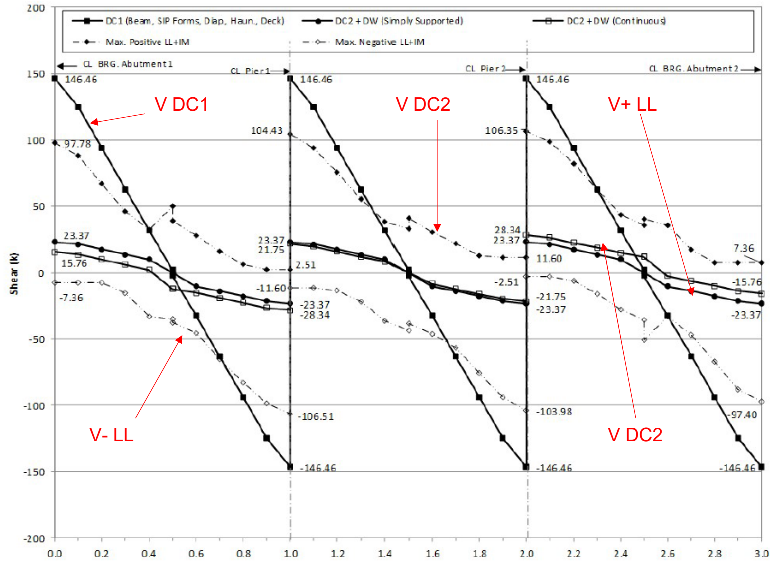


2D Plate and Eccentric Beam Analysis



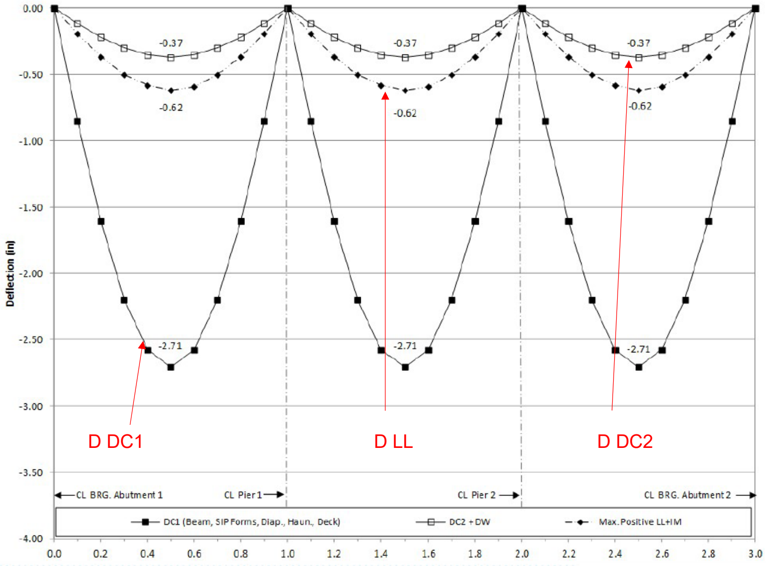
Graph. 2D moment diagram and live load envelope for interior girder

2D Plate and Eccentric Beam Analysis



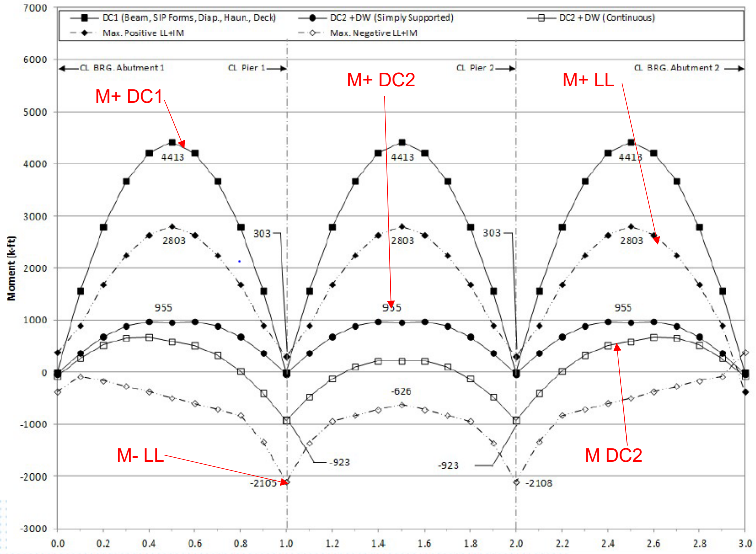
Graph. 2D shear diagram and live load envelope for interior girder

2D Plate and Eccentric Beam Analysis



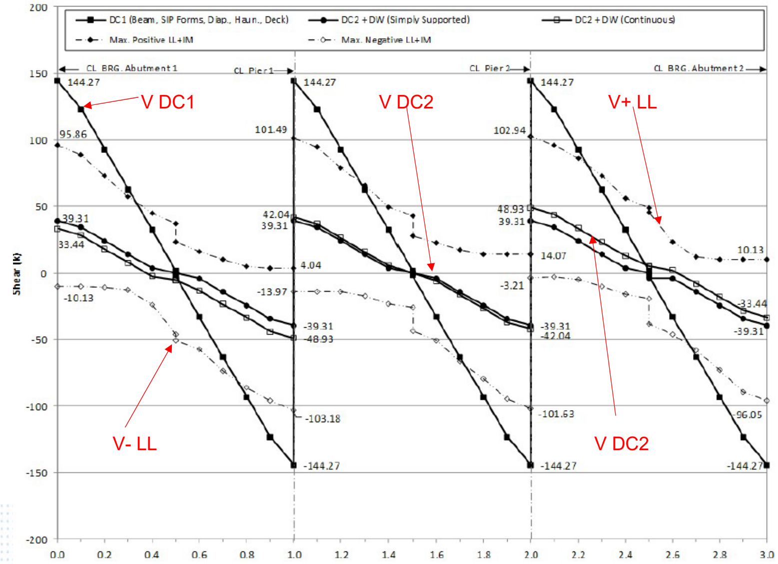
Graph. 2D deflection diagram for interior girder.

2D Plate and Eccentric Beam Analysis



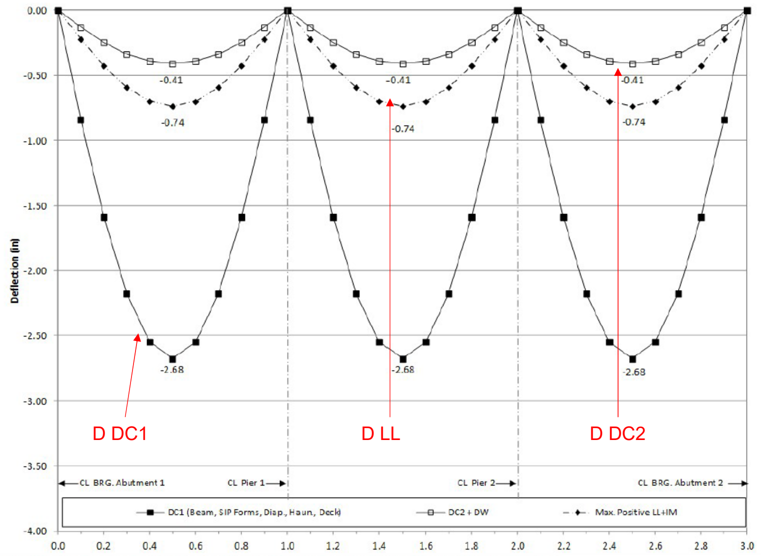
Graph. 2D moment diagram and live load envelope for exterior girder.

2D Plate and Eccentric Beam Analysis



2D shear diagram and live load envelope for exterior girder

2D Plate and Eccentric Beam Analysis

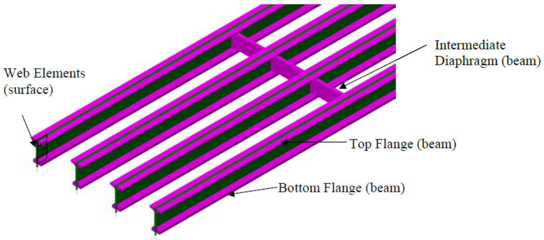


Graph. 2D deflection diagram for exterior girder.

Finite Element Model Analysis

Steps:

1. Create non-composite dead load model
2. Create model for composite dead loads
3. Create a Model for live load
4. Combine analysis results

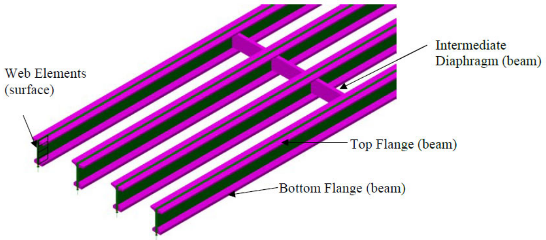


Finite Element Model Analysis

Step 1: Create a model for non-composite

Step 1a – Define Geometry for Girders and Diaphragms

- Girder web surface elements - thick shells
- Flanges and intermediate diaphragms - Beam elements



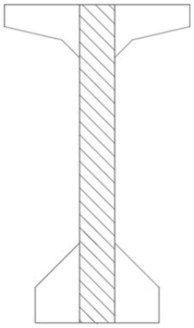


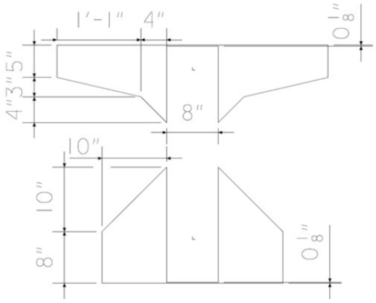
Illustration. Modeling of web depth.

Finite Element Model Analysis

Step 1b – Define Cross-Section Properties

Section Property	Top Flange	Bottom Flange
Cross-section Area (A) (ft^2)	1.736	1.813
Strong Axis Moment of Inertia (I_{yy}) (ft^4)	0.075	0.230
Weak Axis Moment of Inertia (I_{zz}) (ft^4)	1.848	0.976
Torsion Constant (J_{xx}) (ft^4)	0.139	0.195
Shear Area in y direction (A_{vy}) (ft^2)	0.084	0.039
Shear Area in z direction (A_{vz}) (ft^2)	1.544	1.615
Offset in z direction (R_z) (ft)	0.323	-0.568

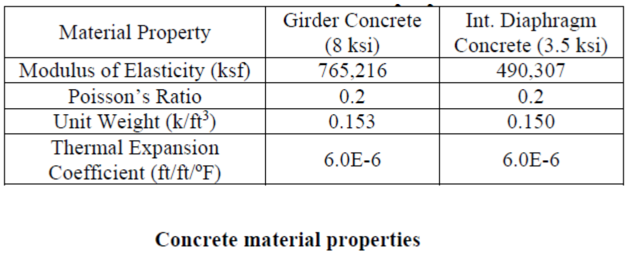
Girder flange section properties.



Girder flanges for 3D FEA model.

Finite Element Model Analysis

Step 1c – Define Material Properties



Material Property	Girder Concrete (8 ksi)	Int. Diaphragm Concrete (3.5 ksi)
Modulus of Elasticity (ksf)	765,216	490,307
Poisson's Ratio	0.2	0.2
Unit Weight (k/ft^3)	0.153	0.150
Thermal Expansion Coefficient (ft/ft/°F)	6.0E-6	6.0E-6

Concrete material properties

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Finite Element Model Analysis

Step 1d – Define Support Conditions

- Simply supported
- One support of each girder is restrained vertically and transversely, other end is restrained vertically, transversely, and longitudinally

Step 1e – Define Non-Composite Loads

- Self-weight & non composite dead loads(SIP forms, haunches, and deck slab)

Step 1f – Define Load Cases- Combined into single load case

Step 1g – Ensure Correct Attributes Are Assigned to Components

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Finite Element Model Analysis

Girders

Web

☐ Thick Shell surface elements

☐ Geometric surface cross-section

☐ Concrete material properties, f'c = 8 ksi

☐ Body force acceleration (gravity)

Flanges

☐ Thick beam elements

☐ Flange cross-sections

☐ Concrete material properties, f'c = 8 ksi

☐ Body force acceleration (gravity)

☐ Other non-composite loads (to top flange only)

Intermediate Diaphragms

Thick beam elements

Geometric cross-section

Concrete material properties f'c = 3.5 ksi

Body force acceleration (gravity)

Step 1h – Run Analysis and Verify Results

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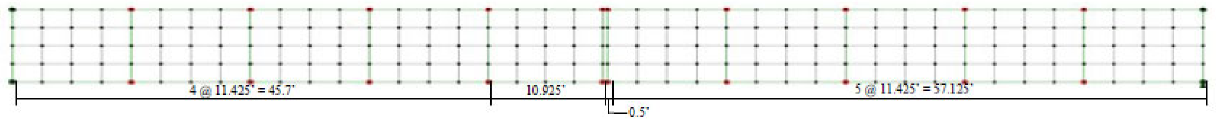
Finite Element Model Analysis

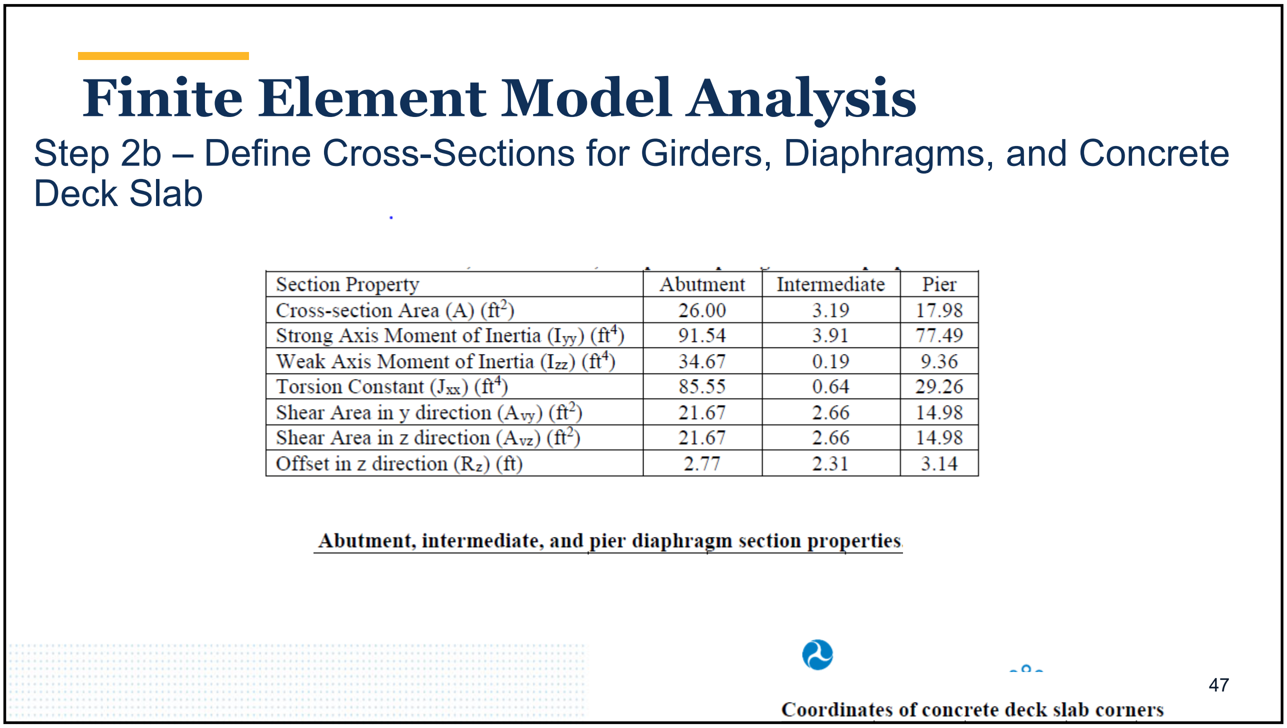
Step 2 – Create Composite Dead Load Model

Step 2a – Define Girder, Diaphragm, and Concrete Deck Slab



Surface definitions for Span 1 of composite model.

Coordinates of concrete deck slab corners



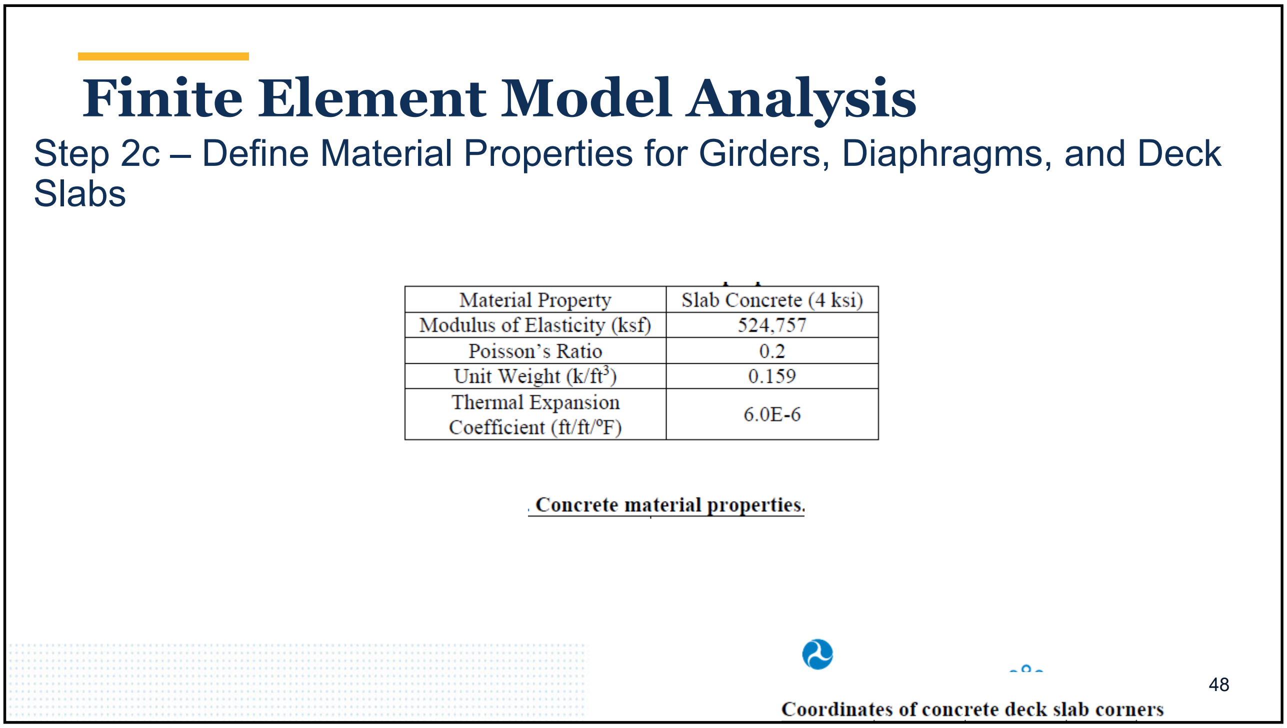
Finite Element Model Analysis

Step 2b – Define Cross-Sections for Girders, Diaphragms, and Concrete Deck Slab

Section Property	Abutment	Intermediate	Pier
Cross-section Area (A) (ft^2)	26.00	3.19	17.98
Strong Axis Moment of Inertia (I_{yy}) (ft^4)	91.54	3.91	77.49
Weak Axis Moment of Inertia (I_{zz}) (ft^4)	34.67	0.19	9.36
Torsion Constant (J_{xx}) (ft^4)	85.55	0.64	29.26
Shear Area in y direction (A_{vy}) (ft^2)	21.67	2.66	14.98
Shear Area in z direction (A_{vz}) (ft^2)	21.67	2.66	14.98
Offset in z direction (R_z) (ft)	2.77	2.31	3.14

Abutment, intermediate, and pier diaphragm section properties

Coordinates of concrete deck slab corners



Finite Element Model Analysis

Step 2c – Define Material Properties for Girders, Diaphragms, and Deck Slabs

Material Property	Slab Concrete (4 ksi)
Modulus of Elasticity (ksf)	524,757
Poisson's Ratio	0.2
Unit Weight (k/ft^3)	0.159
Thermal Expansion Coefficient (ft/ft/°F)	6.0E-6

Concrete material properties.

Coordinates of concrete deck slab corners

Finite Element Model Analysis

Step 2d – Define Support Conditions

- Simply supported made continuous
- One support of each girder is restrained vertically and transversely, other end is restrained vertically, transversely, and longitudinally

Step 2e – Define Loads Applied to Composite Structure

- FWS

Step 2f – Define applied loads for composite structure

Step 2g – Ensure Correct Attributes Are Assigned to Components

Girders

Web

- ☐ Thick Shell surface elements
- ☐ Geometric surface cross-section
- ☐ Concrete material properties, f'c = 8 ksi

Flanges

- ☐ Thick beam elements
- ☐ Flange cross-sections properties
- ☐ Concrete material properties, f'c = 8 ksi

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Finite Element Model Analysis

Abutment intermediate and pier diaphragm

- Geometric cross-section properties
- Thick beam elements
- Concrete material properties, in this example, f'c = 3.5 ksi

Concrete Deck Slab Thick shell surface elements

- Geometric surface cross-section
- Concrete material properties, in this example, f'c = 4 ksi
- FWS and barrier loading

- Step 1h – Run Analysis and Verify Results

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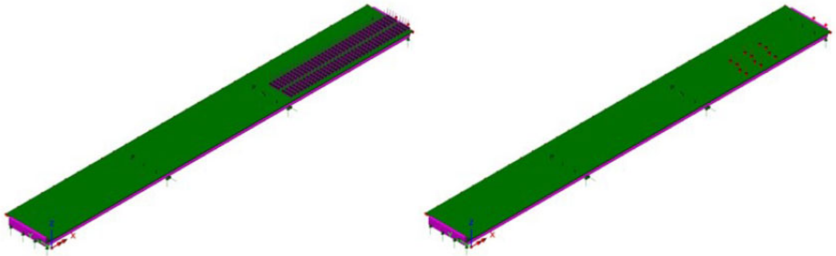
Finite Element Model Analysis

Step 3 – Create Composite Live Load Model

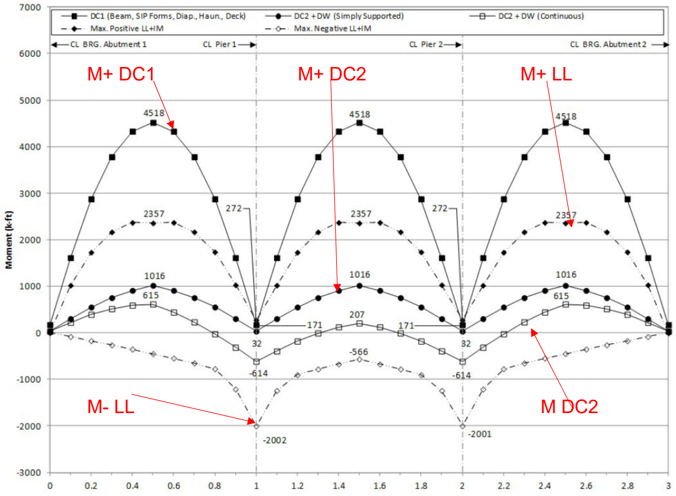
Step 3a – Determine Optimized Live Load

Step 3b – Run Analysis and Verify Results using Simplified Methods

Step 4 – Combine Analysis Results

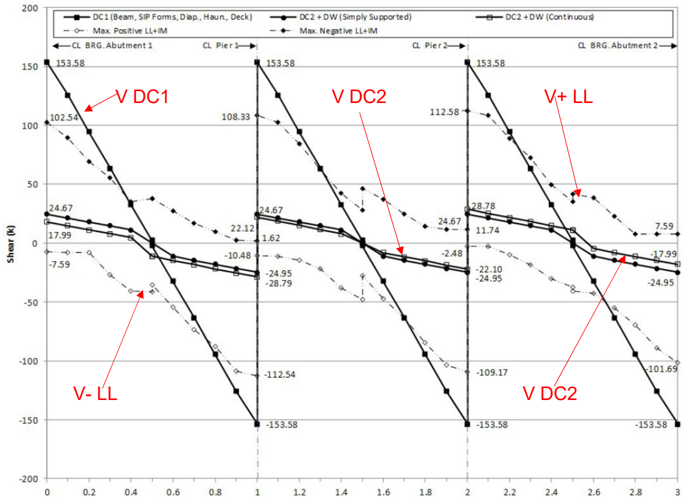


Finite Element Model Analysis



Graph. 3D moment diagram and live load envelope for interior girder.

Finite Element Model Analysis



3D shear diagram and live load envelope for interior girder.

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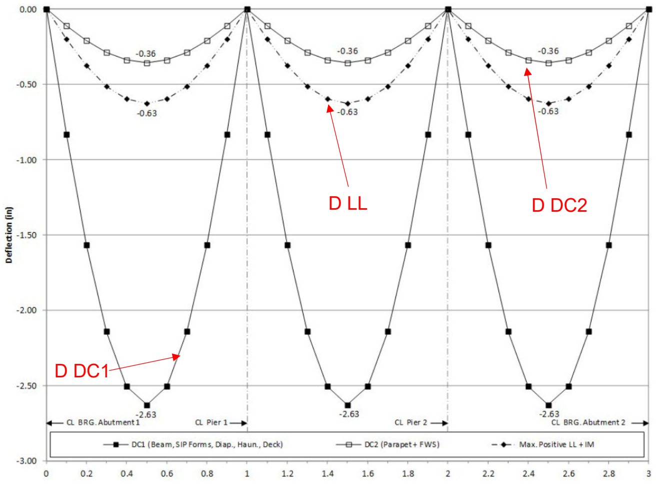
Finite Element Model Analysis



3D deflection diagram for interior girder.

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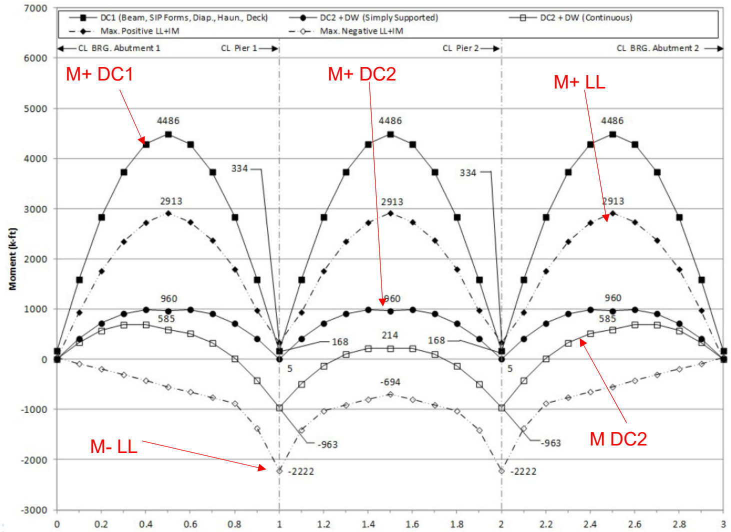
Finite Element Model Analysis



3D moment diagram and live load envelope for exterior girder.

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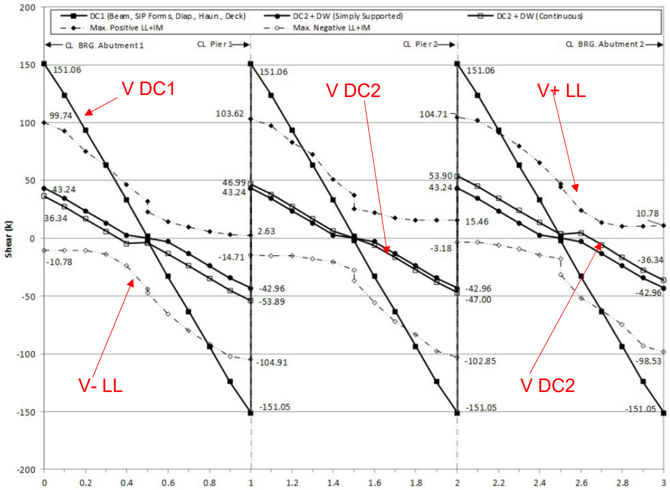
Finite Element Model Analysis



3D shear diagram and live load envelope for exterior girder.

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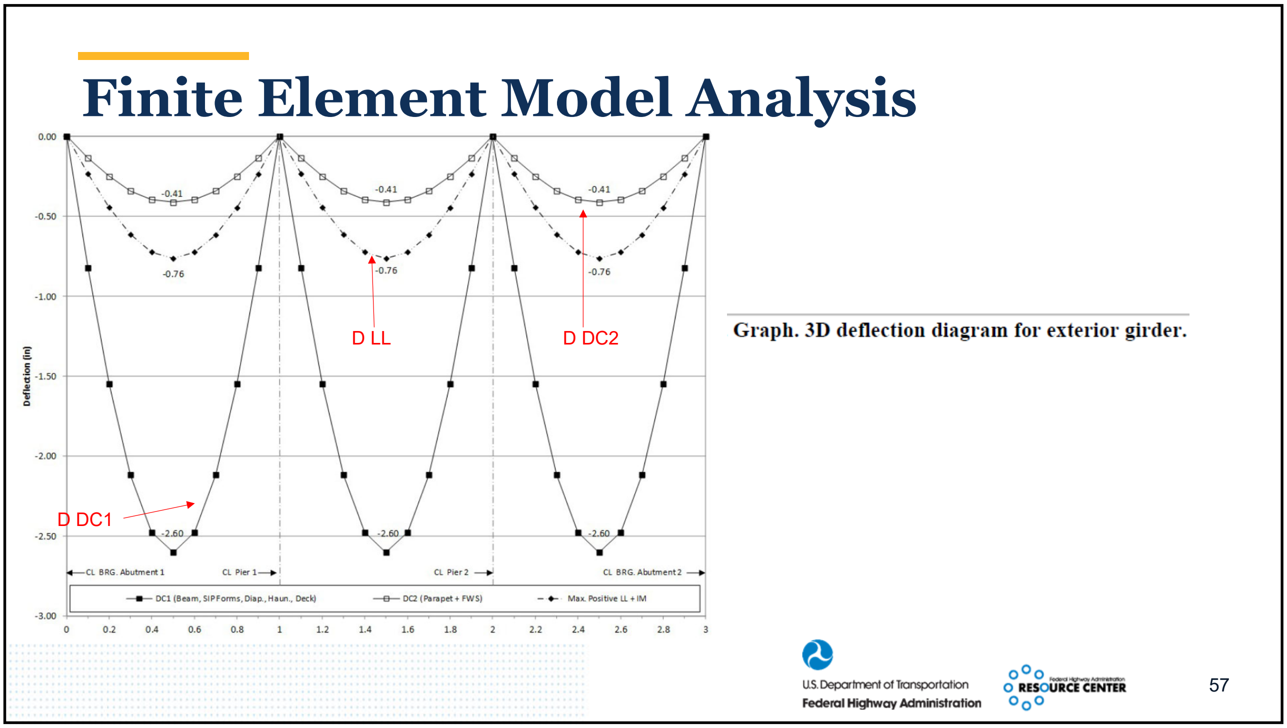

Finite Element Model Analysis

Graph. 3D deflection diagram for exterior girder.

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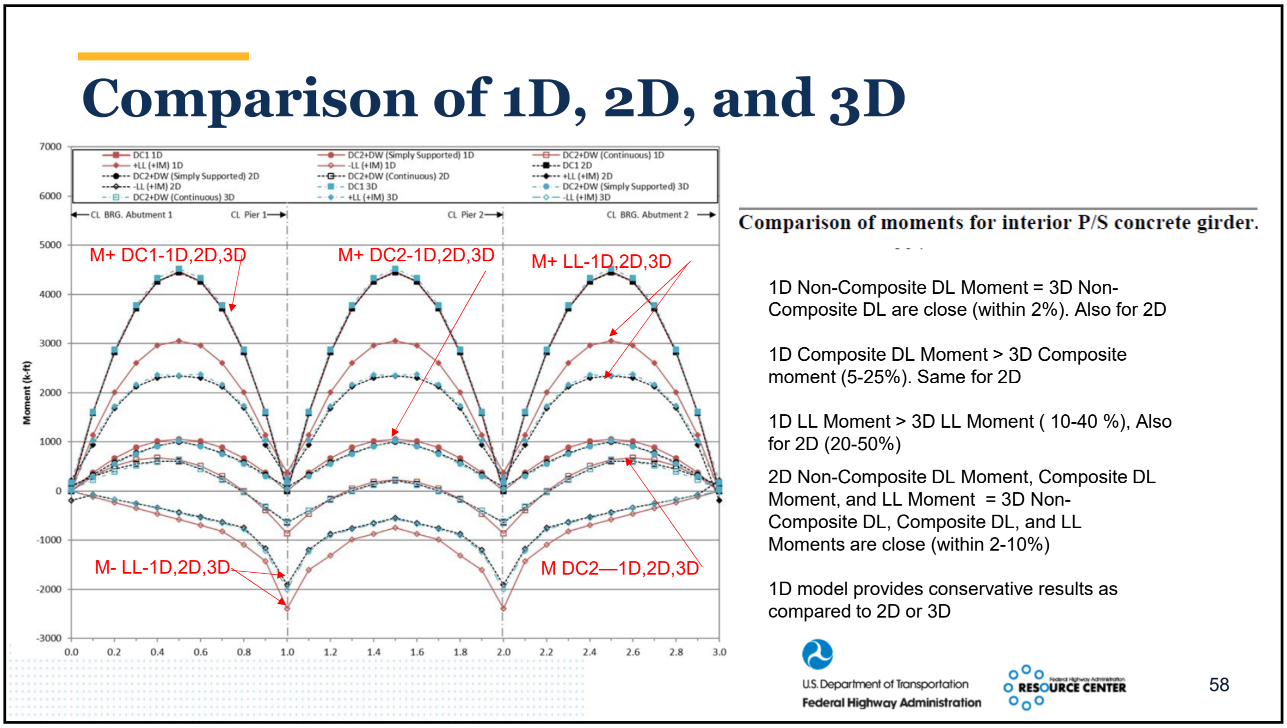

Comparison of 1D, 2D, and 3D

Comparison of moments for interior P/S concrete girder.

1D Non-Composite DL Moment = 3D Non-Composite DL are close (within 2%). Also for 2D

1D Composite DL Moment > 3D Composite moment (5-25%). Same for 2D

1D LL Moment > 3D LL Moment (10-40 %), Also for 2D (20-50%)

2D Non-Composite DL Moment, Composite DL Moment, and LL Moment = 3D Non-Composite DL, Composite DL, and LL Moments are close (within 2-10%)

1D model provides conservative results as compared to 2D or 3D

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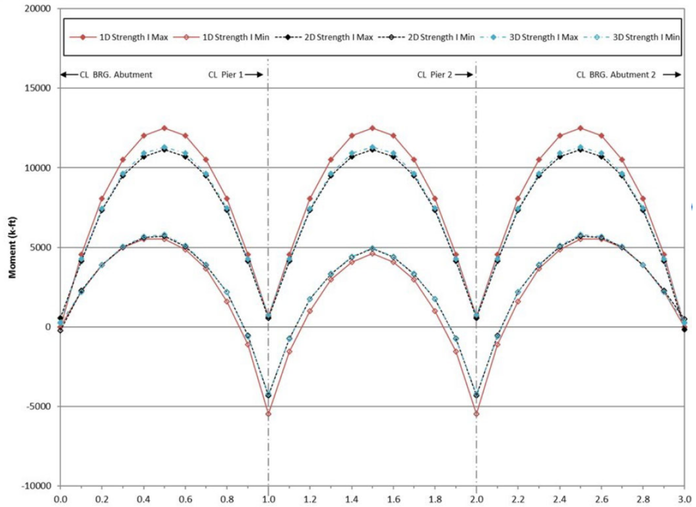
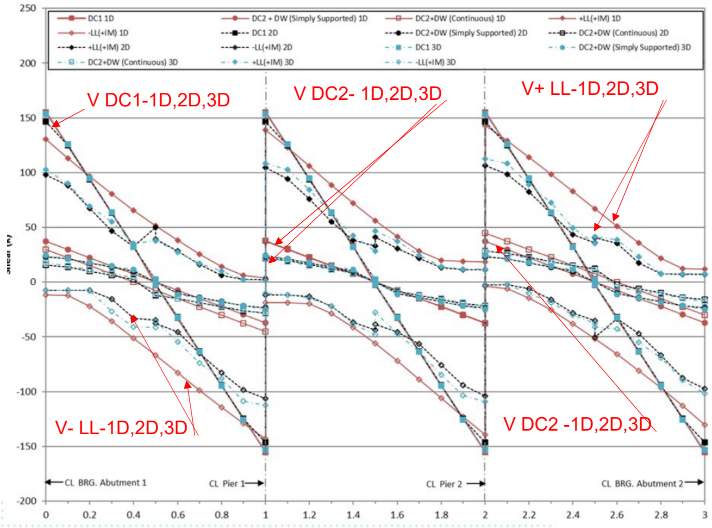
Comparison of 1D, 2D, and 3D



Graph. Comparison of factored moments for interior P/S concrete girder.

Comparison of 1D, 2D, and 3D



V DC1-1D,2D,3D

V DC2- 1D,2D,3D

V+ LL-1D,2D,3D

V- LL-1D,2D,3D

V DC2 -1D,2D,3D

Comparison of shears for interior P/S concrete girder.

1D Non-Composite DL Moment = 3D Non-Composite DL are close (within 2%). Also for 2D

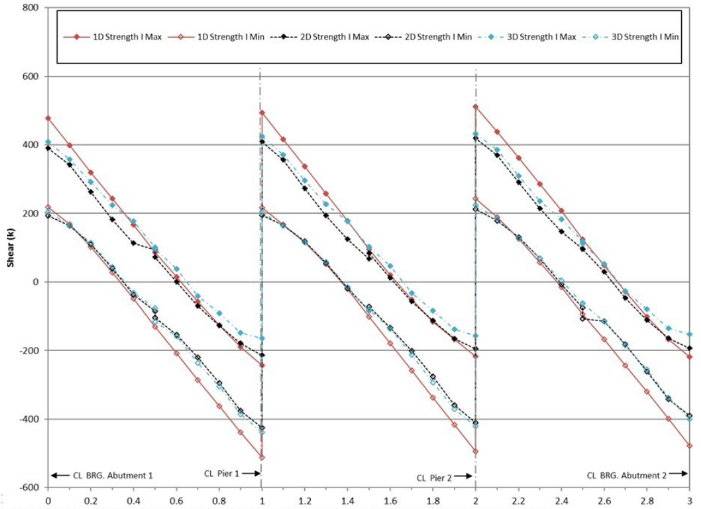
1D Composite DL Moment > 3D Composite moment (0-50%). For 2D (60- 75%) larger at supports

1D LL Moment > 3D LL Moment (inconsistent)

2D Non-Composite DL Moment, Composite DL Moment, and LL Moment = 3D Non-Composite DL, Composite DL, and LL Moments are close (within 1-10%)

1D model provides conservative results as compared to 2D or 3D

Comparison of 1D, 2D, and 3D



Comparison of factored shears for interior P/S concrete girder.

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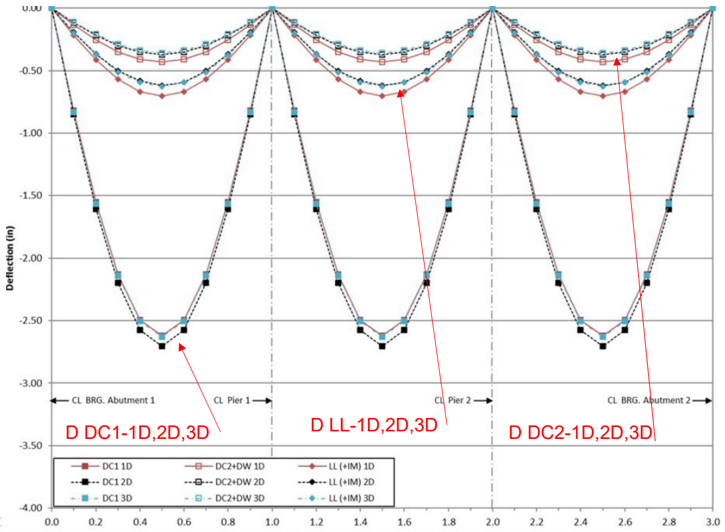
Comparison of 1D, 2D, and 3D



D DC1-1D,2D,3D

D LL-1D,2D,3D

D DC2-1D,2D,3D

Comparison of deflections for interior P/S concrete girder.

1D Non-Composite DL deflection = 3D Non-Composite DL are close (within 2%). Also for 2D (4%)

1D Composite DL deflection > 3D Composite deflection (20-22%). For 2D (15-17%)

1D LL deflection> 3D LL deflection (10%). For 2D (12-14%)

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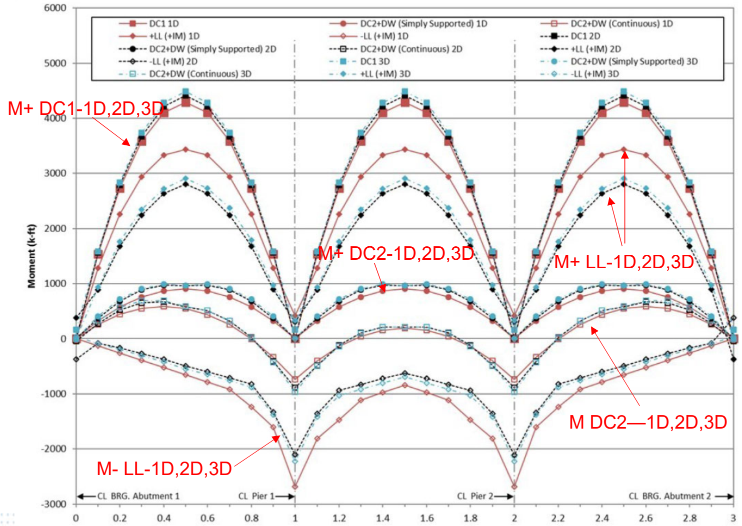
Comparison of 1D, 2D, and 3D



M+ DC1-1D,2D,3D

M+ DC2-1D,2D,3D

M+ LL-1D,2D,3D

M DC2—1D,2D,3D

M- LL-1D,2D,3D

Graph. Comparison of moments for exterior P/S concrete girder.

1D Non-Composite DL Moment < 3D Non-Composite DL are close (3-5 %). Also for 2D

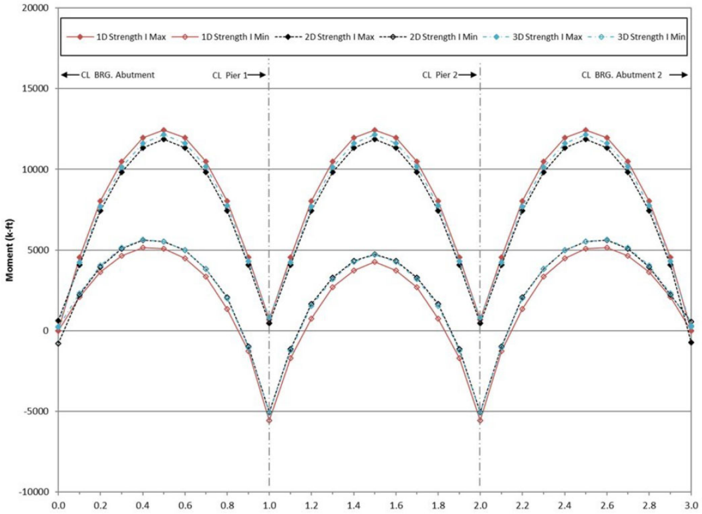
1D Composite DL Moment < 3D Composite moment (5-20%). Same for 2D (15-20%)

1D LL Moment > 3D LL Moment (10-40 %), Also for 2D (20—50%%)

2D Non-Composite DL Moment, Composite DL Moment, and LL Moment = 3D Non-Composite DL, Composite DL, and LL Moments are close (within 2-10%)

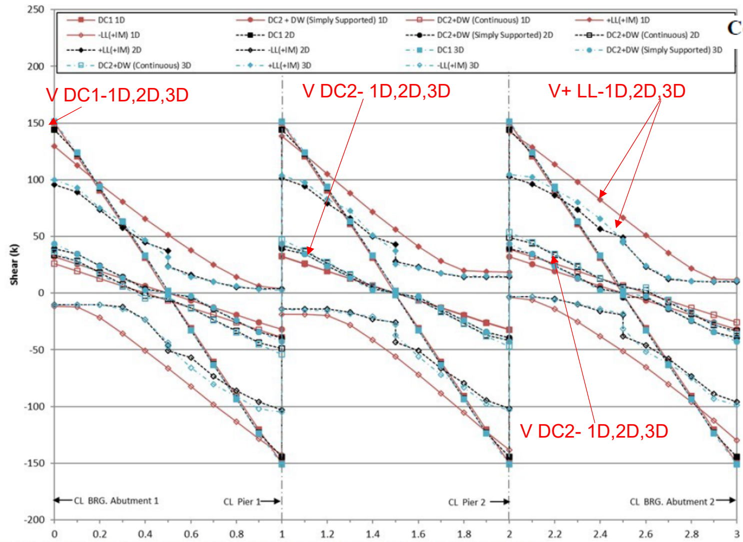
1D model provides conservative results as compared to 2D or 3D

Comparison of 1D, 2D, and 3D



. Comparison of factored moments for exterior P/S concrete girder

Comparison of 1D, 2D, and 3D



Comparison of shears for exterior P/S concrete girder.

1D Non-Composite DL Moment < 3D Non-Composite DL are close (within 5%). Also for 2D

1D Composite DL Moment < 3D Composite moment (20-30%). 2D (60- 75%) larger at supports

1D LL Moment > 3D LL Moment (inconsistent). >2D (35%) at supports

2D Non-Composite DL Moment, Composite DL Moment, and LL Moment < 3D Non-Composite DL, Composite DL, and LL Moments are close (within 0-10%)

1D model provides conservative results as compared to 2D or 3D

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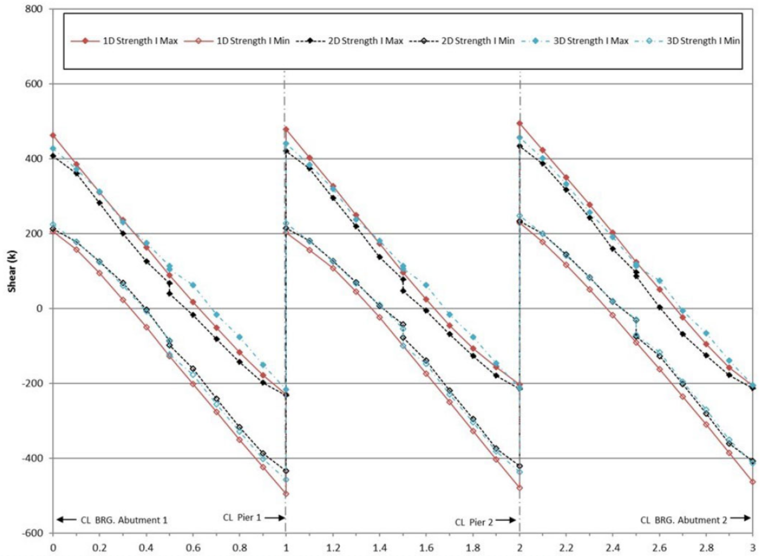
Comparison of 1D, 2D, and 3D



Comparison of factored shears for exterior P/S concrete girder.

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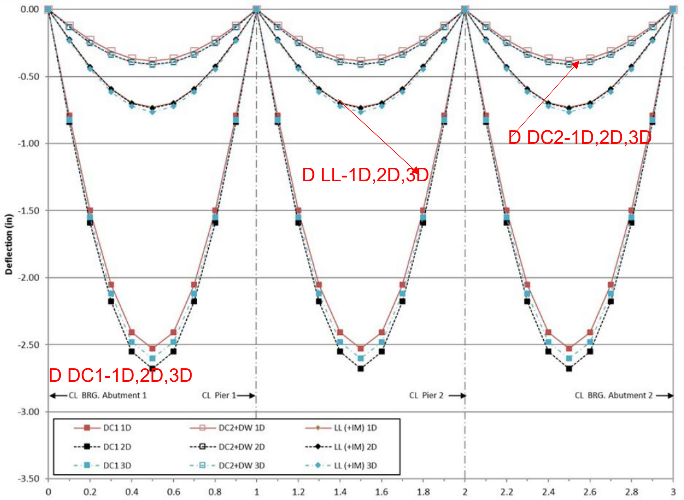
Comparison of 1D, 2D, and 3D



Comparison of deflections for exterior P/S concrete girder.

1D Non-Composite DL deflection < 3D Non-Composite DL are close (within 3%). Also < 2D (6%)

1D Composite DL deflection > 3D Composite deflection (10%). Also < 2D (7-10%)

1D LL deflection< 3D LL deflection (5%). For 2D are equal

2D Non-Composite DL deflection> 3D Non-Composite DL deflection (3%)

2D Composite DL deflection> 3D Composite DL deflection (3%)

2D LL deflection< 3D LL (5%)

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Conclusion

1. 1D line Girder Analysis is adequate for standard straight multi-girder bridges (bread and butter bridges). No lateral analysis

2. 2D Plate and Eccentric Beam (PEB) Analysis is more accurate for skewed and curved bridges. Lateral analysis.

3. Perform 3D Finite Element Analysis to determine web, flanges, diaphragm, and connections behavior and stresses.

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Learning Outcome

1. Perform 1D line Girder Analysis of three span bridge
2. Perform 2D Plate and Eccentric Beam (PEB) Analysis of three span bridge
3. Perform 3D Finite Element Analysis of 3 span bridge
4. Compare results from the three model analysis

Next:
Lesson 6: Verification and Validation of Results

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Lesson 6 – Verification and Validation of Results

Learning Outcomes

By the end of this lesson, you should be able to:

- Describe the difference between model verification and validation
- Discuss software verification methods
- Discuss design model verification methods
- Troubleshoot typical modeling problems
- Verify the results of a FEA model thru illustrative examples

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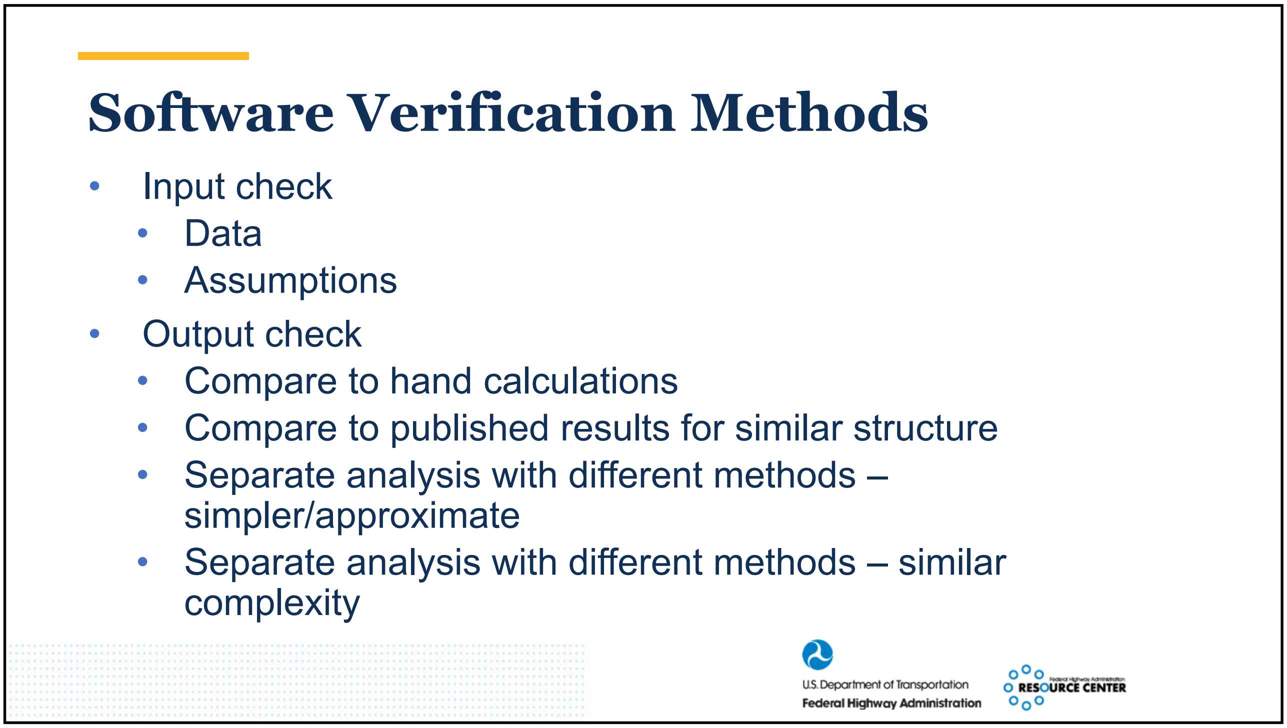
Verification vs. Validation

- **Verification** – Ensures that the model is behaving as intended and giving results that are expected
  - Software Verification – software is appropriate for wider use by the agency or design office
  - Design Model Verification – results from a designer’s model can be used to verify capacity
- **Validation** – Confirming that the model is behaving the same as an actual structure

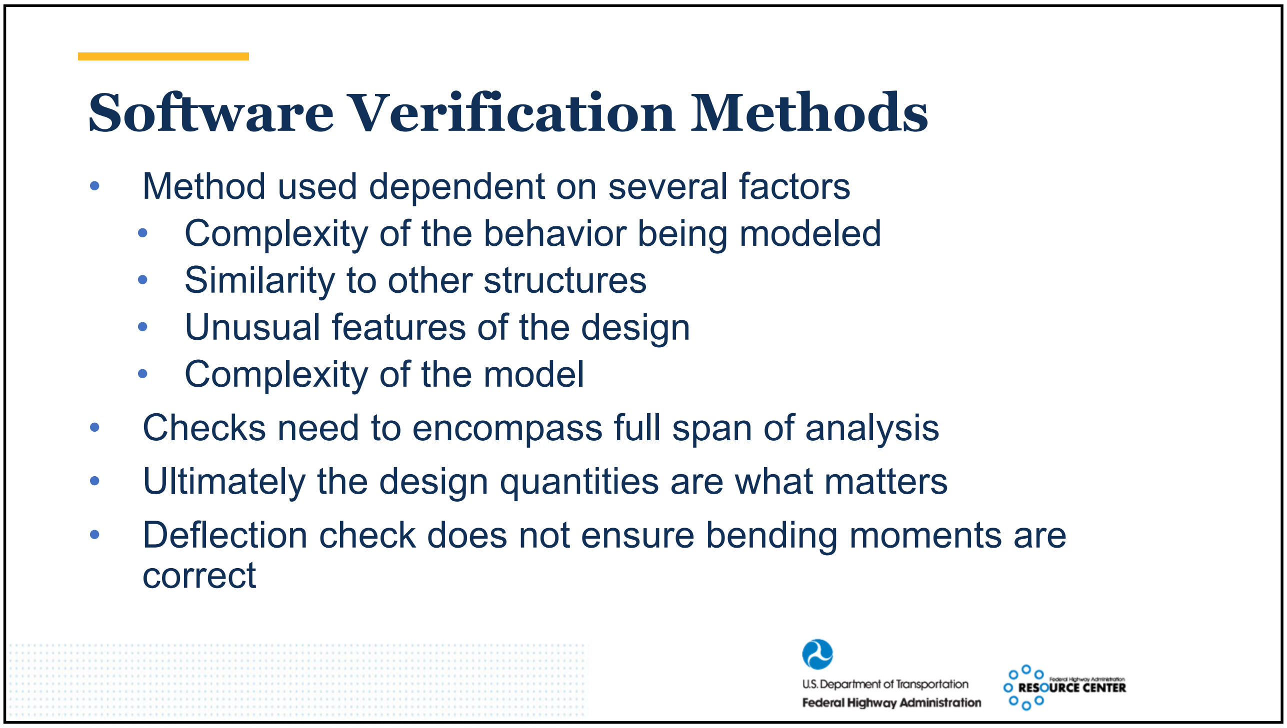
Software Verification Methods

- As part of QA/QC procedures, need to check results of analysis
- Can be challenging for more complex analyses
- Several different options
- “Sanity check”
- Sensitivity study



Software Verification Methods

- Input check
  - Data
  - Assumptions
- Output check
  - Compare to hand calculations
  - Compare to published results for similar structure
  - Separate analysis with different methods – simpler/approximate
  - Separate analysis with different methods – similar complexity



Software Verification Methods

- Method used dependent on several factors
  - Complexity of the behavior being modeled
  - Similarity to other structures
  - Unusual features of the design
  - Complexity of the model
- Checks need to encompass full span of analysis
- Ultimately the design quantities are what matters
- Deflection check does not ensure bending moments are correct

Software Verification Methods

- Compared to published data/tables
  - AISC Moment, Shears, and Reactions for Continuous Highway Bridges
  - Hellmut Homberg charts (Influence surface)
  - Roark’s Formulas for Stress and Strain
  - AISC Steel Construction Manual
  - NSBA LRFD Simon 55
  - Euler buckling

Design Model Verification Methods

- General Checks of the Model
  - Do the total reactions equal the load applied?
  - Are the reactions in the directions expected?
  - Are the reactions distributed as expected?
  - Is the displaced shape continuous?
  - Are the magnitudes of the displacements reasonable?
  - Is everything that should be connected together moving together?

Design Model Verification Methods

- Verifications of a Complex Model
  - Start with a simplified model, check the results with published or hand calculated data
  - Add complexity, check results
  - Example: reduce model to a simply supported beam, check moments, deflections, reactions

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Design Model Verification Methods

- Approximate methods
  - A large variety of methods have been developed in the past to save effort
    - Elastic center method for frames
    - Column analogy for frames
    - Three-moments equation
    - Portal frame method
    - V-Load method for curved bridges
    - NSBA Steel Bridge Design Manual, Chapter 8 (2/22)
- Comparing models built in other packages – e.g. vehicles modeller

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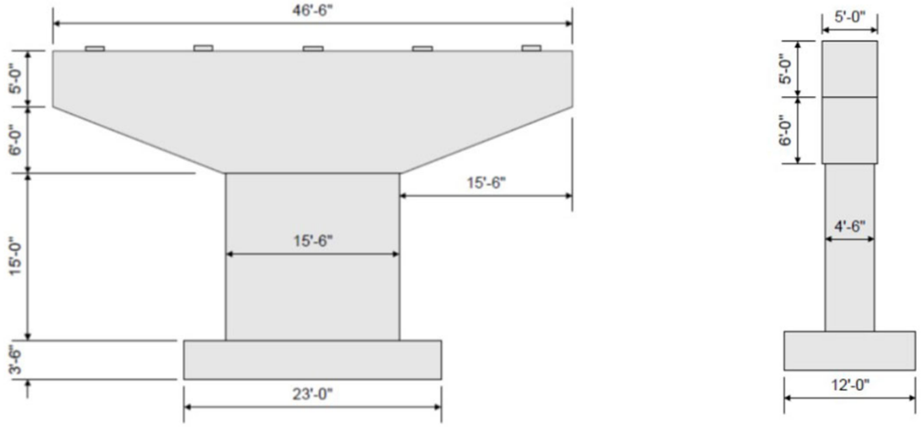
Example 1– Strut & tie analysis of pier cap



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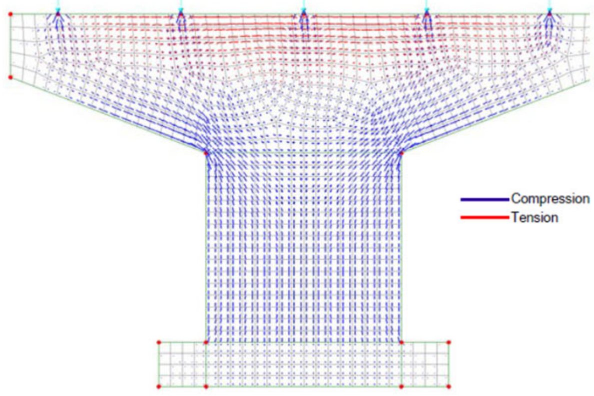
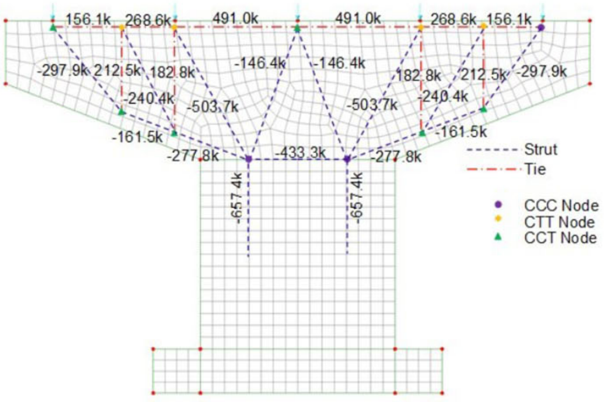
Example 1– Strut & tie analysis of pier cap



Flow of Principal stress field



Struts & ties configuration & results

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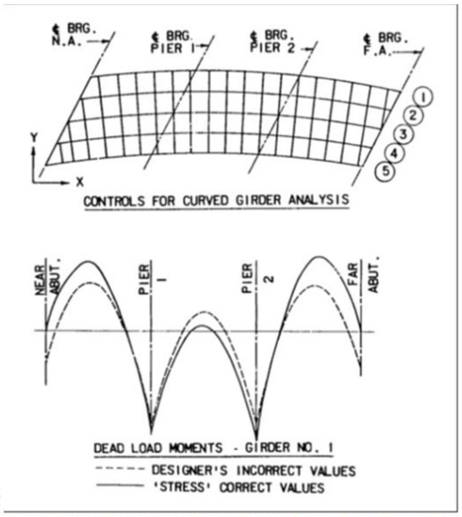
Example 2– Modeling errors ramification



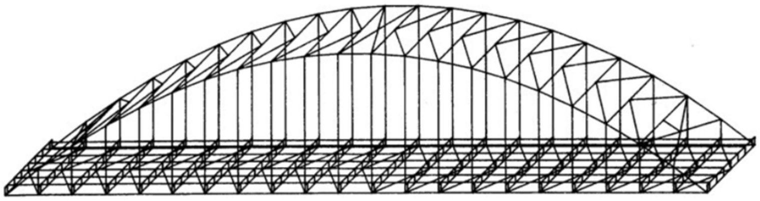
Far abutment vertical reactions for fixed vs free rotational DOT about x-axis

FAR ABUTMENT (FA) SUPPORT REACTIONS				
GIRDER NO.	"GRID" CORRECT SUPPORT CONDITION	DESIGNER'S INCORRECT REACTIONS	"GRID" INCORRECT SUPPORT CONDITIONS USING DESIGNER'S ASSUMPTIONS	
	(VERTICAL – k)	(VERTICAL – k)	(VERTICAL – k)	(MOMENT X k-ft)
1	66.68	223.61	223.24	811.24
2	64.36	47.07	47.14	1231.35
3	64.93	86.29	85.49	1229.66
4	66.62	117.94	117.94	1237.95
5	69.46	-81.63	-81.51	813.37

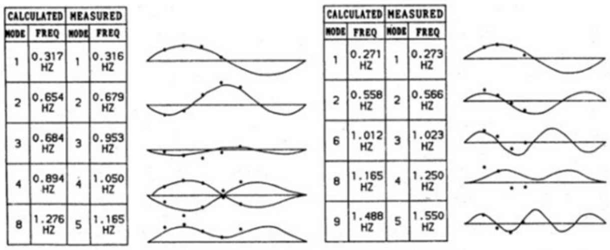
Validating Models of Existing Bridges

- Compare initial results to original designers' calculations.
- May require use of gap & one way elements to account for wear and tear such as pin wear.
- Investigate how the original structure was erected such that dead load analysis assumptions are correct.
- Compare analytically-obtained solutions to field-estimated data or measurements such as displacement, force, mode shapes, strain, etc.

Example 3 – Field validation of an arch truss



Isometric view of three-dimensional finite element model



CALCULATED		MEASURED	
MODE	FREQ	MODE	FREQ
1	0.317 HZ	1	0.316 HZ
2	0.654 HZ	2	0.679 HZ
3	0.684 HZ	3	0.953 HZ
4	0.894 HZ	4	1.050 HZ
8	1.276 HZ	5	1.165 HZ

CALCULATED		MEASURED	
MODE	FREQ	MODE	FREQ
1	0.271 HZ	1	0.273 HZ
2	0.558 HZ	2	0.566 HZ
6	1.012 HZ	3	1.023 HZ
8	1.165 HZ	4	1.250 HZ
9	1.488 HZ	5	1.550 HZ

Comparison of field-measured vs. calculated frequencies and mode shapes

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Polling Question 1 (L6S17Q1):

As the analysis capabilities of a software increase, the need for independent verification decreases.

a) True
b) False

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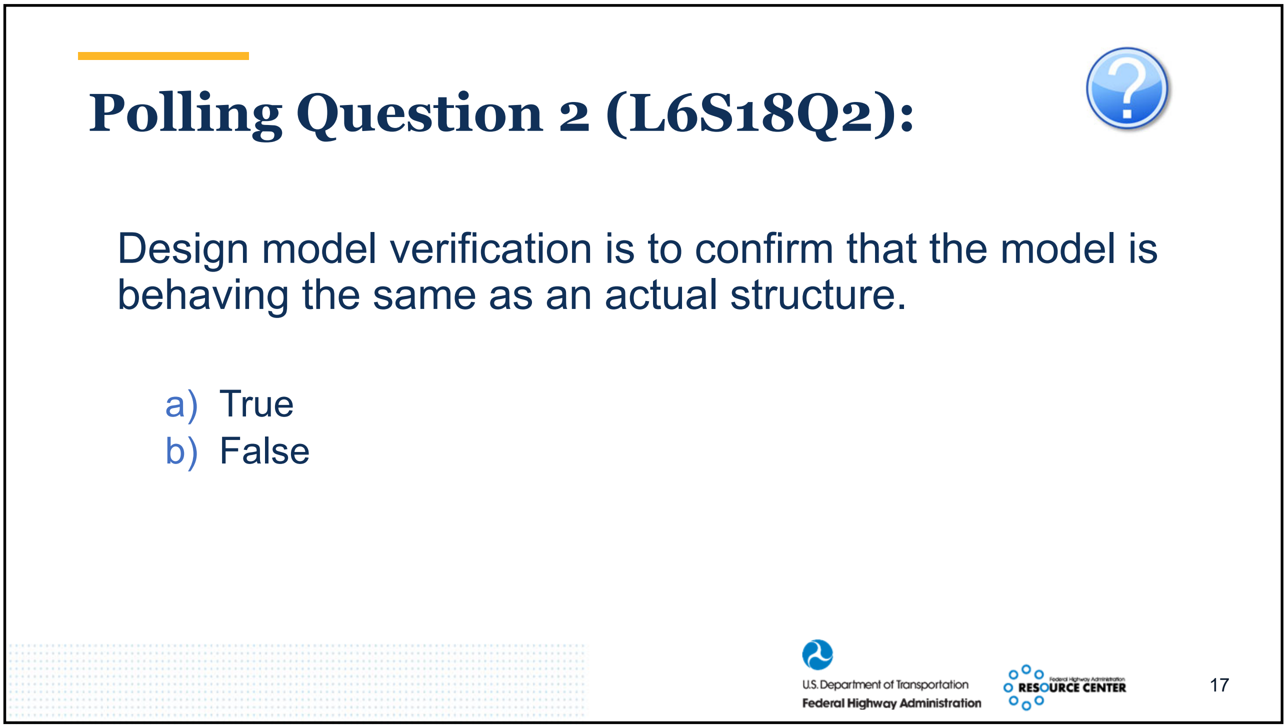
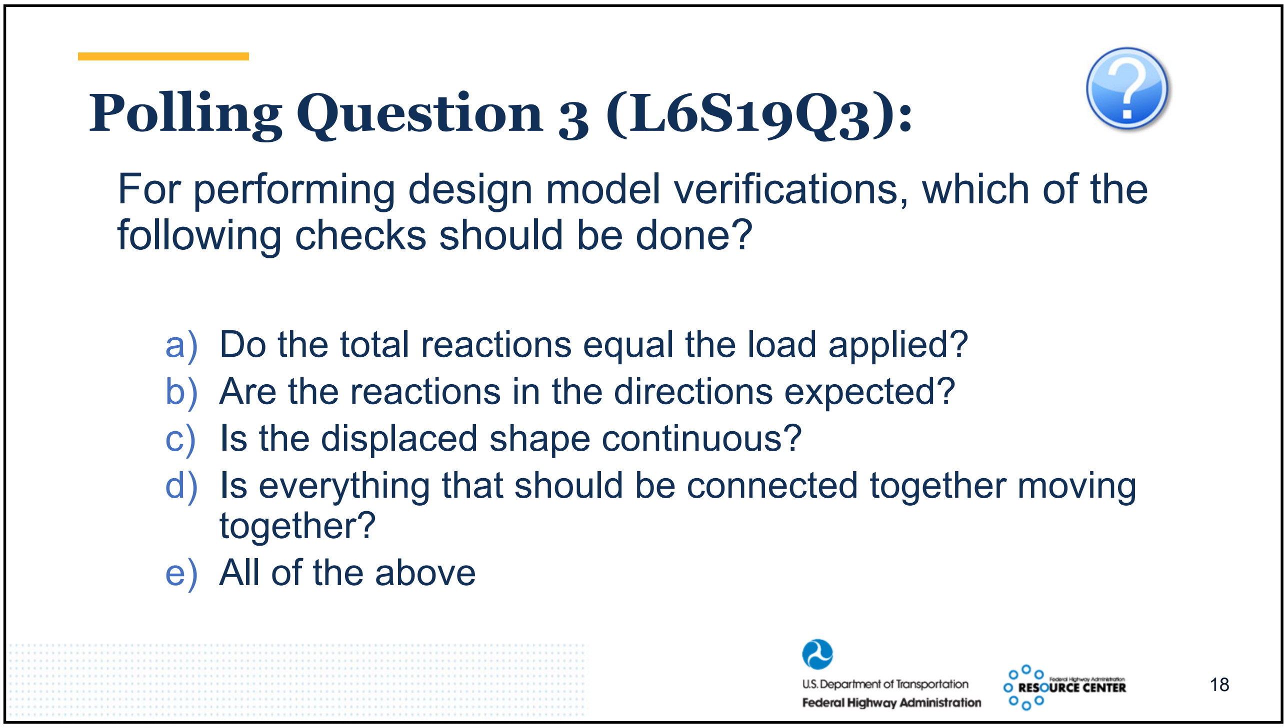


Polling Question 2 (L6S18Q2):

Design model verification is to confirm that the model is behaving the same as an actual structure.

a) True
b) False



Polling Question 3 (L6S19Q3):

For performing design model verifications, which of the following checks should be done?

a) Do the total reactions equal the load applied?
b) Are the reactions in the directions expected?
c) Is the displaced shape continuous?
d) Is everything that should be connected together moving together?
e) All of the above

Learning Outcome Review

- Describe the difference between model verification and validation
- Discuss software verification methods
- Discuss design model verification methods
- Troubleshoot typical modeling problems
- Verify the results of a FEA model thru illustrative examples

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Next:
Lesson 7 Analysis to Design

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Lesson 7 - Analysis to Design

Learning Outcomes

By the end of this lesson, you should be able to:

- Explain the pitfalls of using FEA model results in design
- Process FEA results into design demand forces by integrating stresses
- Evaluate construction interface demand forces by FEA
- Perform cross-frame modeling & design
- Describe segmental bridge analysis and design
- Calculate slab equivalent width considering shear lag effect

LRFD Design

- Strength Limit State
  - Individual member design in the form of factored $M, V, and\ P$ (i.e. force level)
- Service and Fatigue Limit States
  - Normally, check shear & normal stresses independently
  - Except checking principal stress for the web of segmental concrete bridges (AASHTO 5.9.2.3.3)
- Typical FEA model output
  - Typically the output of FEA is in the form of stress at nodes & Gaussian points, or any point if $[B]$ at the point is defined.
  - Need convert from stress level to force level for design

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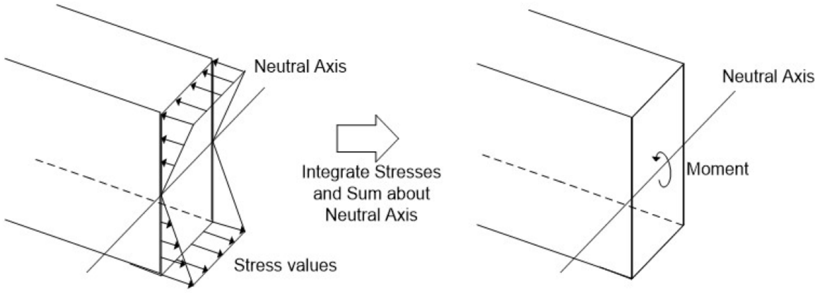
Model Post-Processing
(FHWA Manual 8.2.1)

- Integrating stresses to determine member demand forces
- FE software provides stress values at nodal and Gaussian points



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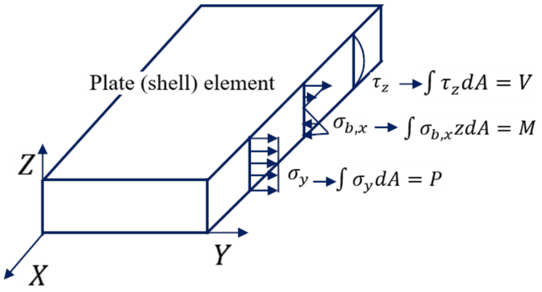
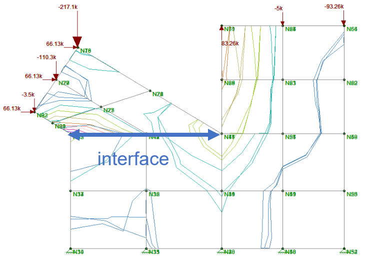
Example - Deck Shell Element

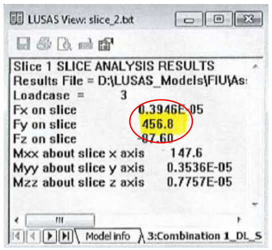
- Demand axial load (P) :
  - $\int \sigma_y dA = P$
- Demand moment (M):
  - $\int \sigma_{b,x} z dA = M$
- Demand shear force (V):
  - $\int \tau_z dA = V$
- Some FEA software have integrating stresses utility to provide $M, V, and\ P$ for design



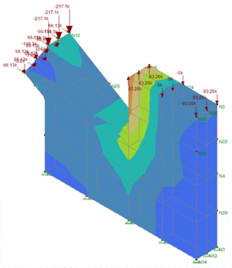
Example – Find the construction interface shear & compression forces

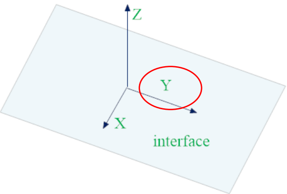


```
Slice 1 SLICE PROPERTIES
Total Slice Area             13.46
Ixx about slice x axis       66.30
Iyy about slice y axis       3.434
Ixy about origin           0.000
Global X of centroid       0.3802E-13
Global Y of centroid        14.94
Global Z of centroid        36.48
Principal I1  at centroid   66.30
Principal I2 at centroid    3.434
Principal Angle (degrees)    90.00
```



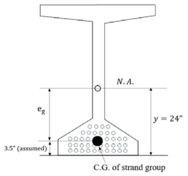
```
Slice 1 SLICE ANALYSIS RESULTS
Results File = D:\LUSAS_Models\FIU\As:
Loadcase  =       3
Fx on slice              0.3946E-05
Fy on slice              456.8
Fz on slice              -87.60
Mxx about slice x axis      147.6
Myy about slice y axis     0.3536E-05
Mzz about slice z axis     0.7757E-05
```

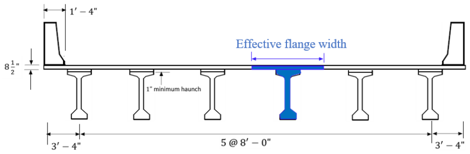


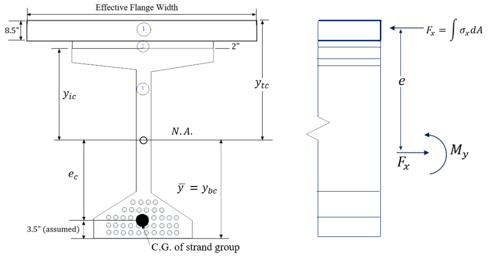


Concrete Girder Analysis to design

- Concrete girders – using PEB 2-D model
  - Non-composite DL1:
    - Demand M_{DL1} is directly from beam element
  - Composite DL2 (SBC, FWS):
    - Demand M_{DL2}, is obtained by summing the moments about the centroid of the composite section
    - $M_{DL2} = M_y + F_x e$







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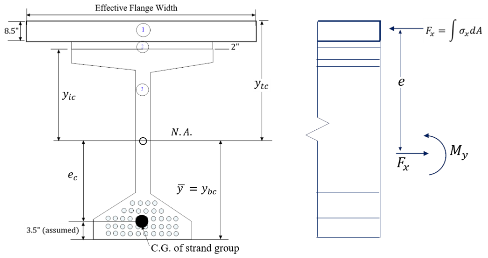
Concrete Girder Analysis to design

- Concrete girders – using PEB 2-D model
  - Composite LL:
    - Demand M_{LL}, is obtained by summing the moments about the centroid of the composite section
    - $M_{LL} = M_y + F_x e$



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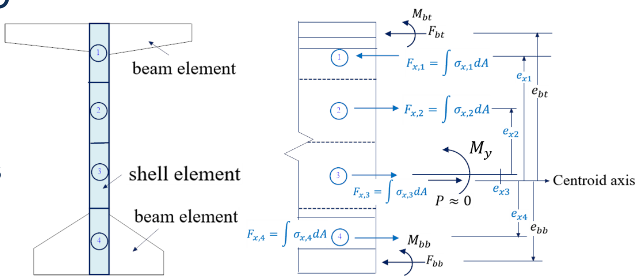
Concrete Girder Analysis to design

- Concrete girders – using 3D FEA model
- Non-composite DL1:
  - Demand M_{DL1} is from integration stress at web plus moment due to flange beam elements

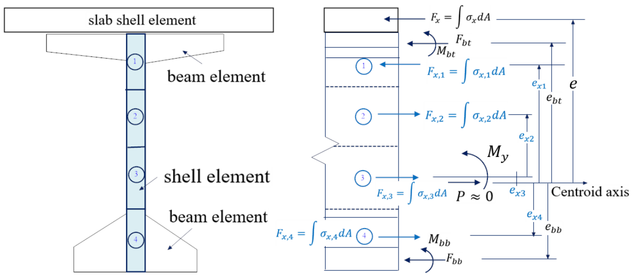


- $M_{DL1} = (M_{bt} + F_{bt} \times e_{bt}) + ((M_{bb} + F_{bb} \times e_{bb})$

  $+F_{x,1} \times e_{x1} + F_{x,2} \times e_{x2}$
  $+F_{x,3} \times e_{x3} + F_{x,4} \times e_{x4}$
- $P = \sum F_x \cong 0$

Concrete Girder Analysis to design

- Concrete girders – using 3D FEA model
  - Composite DL2 (SBC, FWS):
  - Demand M_{DL2} is from integration stress at web + flange beam M's + integration stress at slab



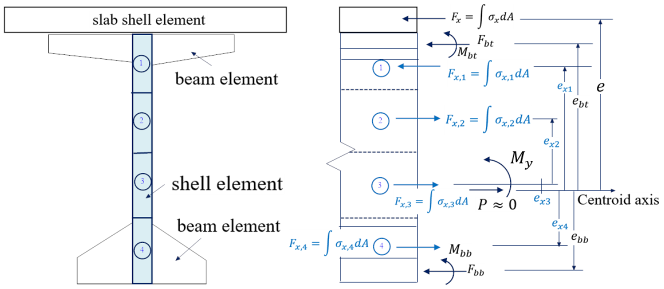
- $M_{DL2} = (M_{bt} + F_{bt} \times e_{bt}) + ((M_{bb} + F_{bb} \times e_{bb})$

  $+F_{x,1} \times e_{x1} + F_{x,2} \times e_{x2}$
  $+F_{x,3} \times e_{x3} + F_{x,4} \times e_{x4}$

  $+F_x \times e$
- $P = \sum F_x \cong 0$

Concrete Girder Analysis to design

- Concrete girders – using 3D FEA model
  - Composite LL:
  - Demand M_{LL} is from integration stress at web + flange beam M's + integration stress at slab



- $M_{LL} = (M_{bt}+F_{bt} \times e_{bt}) + \big((M_{bb}+F_{bb} \times e_{bb}\big)$

  $+F_{x,1} \times e_{x1} + F_{x,2} \times e_{x2}$
  $+F_{x,3} \times e_{x3} + F_{x,4} \times e_{x4}$

  $+F_x \times e$
- $P = \sum F_x \cong 0$

Concrete Girder Analysis to design

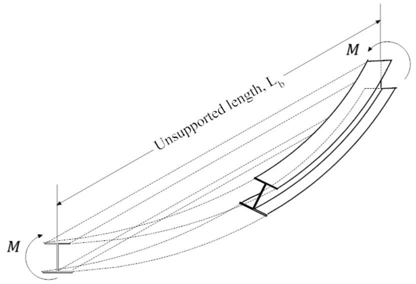
- Follow AASHTO LRFD Design:
- SERVICE I & III:
  - Stress (compression & tension) level check (AASHTO 5.9.4.2)
- STRENGTH I:
  - Force level check (AASHTO 5.7.3)
  - $\phi M_n \geq M_u$
  - $M_u = 1.25[M_{DL1}] + 1.5M_{DL2} + 1.75M_{LL+IM}$
  - $M_n = A_{ps} f_{ps} \left(d_p - \frac{a}{2}\right) + 0.85 f_c' (b - b_w) h_f \left(\frac{a}{2} - \frac{h_f}{2}\right)$ (AASHTO 5.7.3,2.2-1)

Steel Girder Analysis to design

- Demand forces calculation –
  - similar to concrete girder with consideration of DL1, DL2, LL, etc.
- Design – follow AASHTO LRFD design specs 6.10.
  - SERVICE II: $DL1 + DL2 + 1.3(LL + IM)$ with $IM = 0.33$
    - Stress level check (AASHTO 6.10.4)
  - FATIGUE I: $1.75(LL + IM)$ with $IM = 0.15$
    - Stress level check (AASHTO 6.10.5)
  - STRENGTH I: $1.25\ DL1 + 1.5DL2 + 1.75(LL + IM)$
    - Force level check (AASHTO 6.10.6)
    - $\phi M_n \geq M_u$

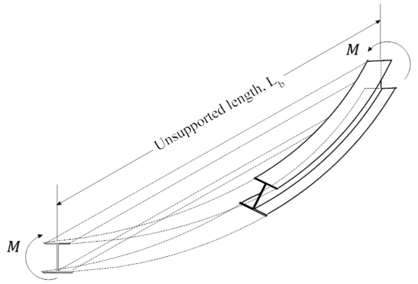
Steel Girder Design

- Lateral torsional buckling (AASHTO 6.10.1)
  - girder may buckle globally when subjected to moments and/or axial load if the unsupported length is too long.
  - Intermediate cross frames provide bracing against lateral torsional buckling of girder during erection and concrete deck placement.



Steel Girder Design

- Lateral torsional buckling (AASHTO 6.10.1)
  - Intermediate cross frames are critical while the girders are in the non-composite stage under wind loading.



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Steel Girder Design

- Local buckling (AASHTO 6.10.8)

As shown in Figure E.4, When the width-to-thickness ratio of a plate is getting larger, the plate may buckle locally when subjected to compression load.

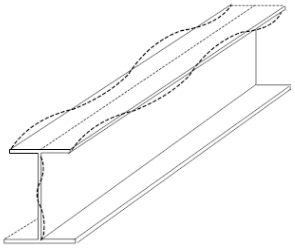

Figure E.4 Local Buckling of plates.

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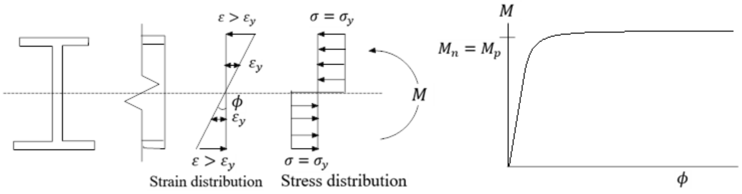
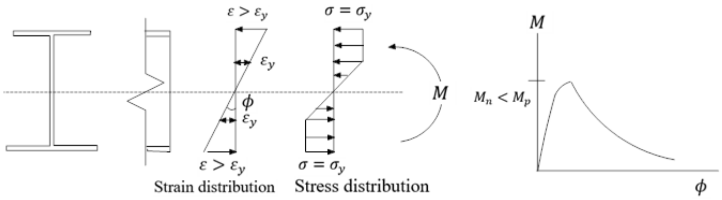
Steel Girder Design

- Compact & non-compact sections (AASHTO 6.10.7)
  - Compact section can develop full plastic moment capacity, M_p at which the whole cross section reaches the yielding stress, σ_y.



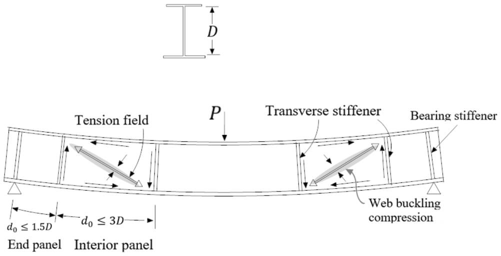
Steel Girder Design

- Compact & non-compact sections (AASHTO 6.10.7)
  - A non-compact section can only develop partial yielding stress distribution along the cross section before compressive flange plate local buckling occurs.



Steel Girder Design

- Tension field action (AASHTO 6.10.9)
  - Two transverse stiffeners can serve as anchors for the tension field force developed in the interior shear panel of the web, if web local buckling occurs within the panel.
  - Tension field action allows the web developing its post buckling capacity without suddenly losing the shear capacity.
  - Tension field action can be imaged as a laterally loaded cable to prevent large lateral web displacement.



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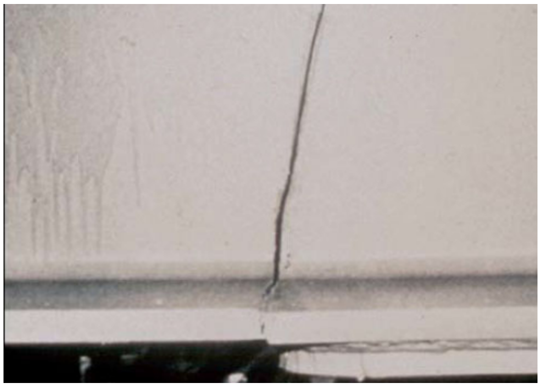
Steel Girder Design

- Fatigue Consideration (AASHTO 6.6)
  - Fatigue crack can form and propagate from weld discontinuities or stress concentration locations if a girder is subjected to significantly cyclic live loads.
  - AASHTO 6.6.2 requires a minimum CVN (Charpy V-notch) toughness test at specified temperature for the plate in tension due to tension force or bending.



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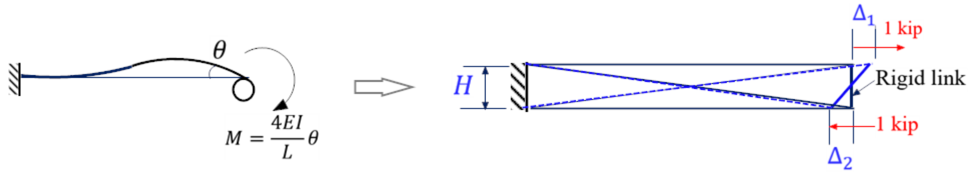
Cross-Frame Analysis to design

- **Recall:** Simply cross-frame to a prismatic member
  - Effective moment of initial of prismatic member for bending, I_b

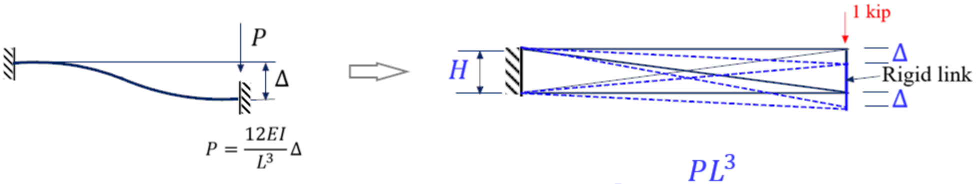


$$M = 1 \times H;\ \theta = \frac{\Delta_1 + \Delta_2}{H}$$

$$I_b = \frac{ML}{4E\theta}$$

Cross-Frame Analysis to design

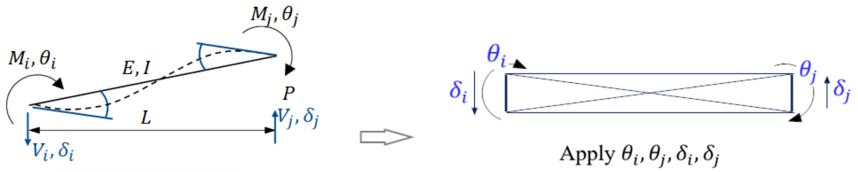
- **Recall:** Simply cross-frame to a prismatic member
  - Effective moment of initial of prismatic member for shear, I_s



$$I_s = \frac{PL^3}{12E\Delta}$$

Cross-Frame Analysis to design

- Obtain $\theta_i, \theta_j, \delta_i, \delta_j$ of prismatic member from structural analysis
- Apply $\theta_i, \theta_j, \delta_i, \delta_j$ to cross frame to get member forces
- Design members per AASHTO



Obtain $\theta_i, \theta_j, \delta_i, \delta_j$
from analysis

Apply $\theta_i, \theta_j, \delta_i, \delta_j$
to cross frame to get
member forces

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Cross-Frame Analysis to design

- Design members per AASHTO 4.5.3
  - Check cross-frame member buckling capacity:
    - If $P < P_{max}$ No buckling occurs
    - If $P > P_{max}$ buckling occurs
    - Elastic buckling (i.e. $\frac{KL}{r} > C_c$): $P_{\max} = \frac{\pi^2 AE}{\left(\frac{KL}{r}\right)^2}$
    - Inelastic buckling (i.e. $\frac{KL}{r} \leq C_c$):
    - $C_c = \sqrt{(2\pi^2 E)/\sigma_y}$

$$P_{\max} = \left[1 - \frac{\left(\frac{KL}{r}\right)^2}{2C_c^2}\right]\sigma_y A$$

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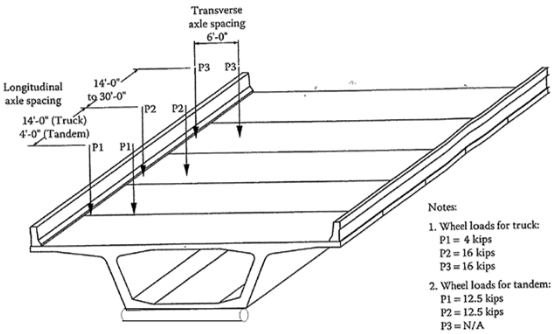
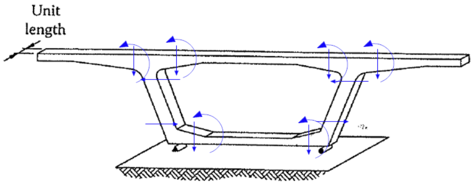
Segmental Box Girder Analysis to design

- Transverse analysis:
  - Use influence surface utility from any FEM software find critical transverse LL moments (k-ft/ft) in slab, webs, and bottom floor.



Notes:

1. Wheel loads for truck:
   P1 = 4 kips
   P2 = 16 kips
   P3 = 16 kips
2. Wheel loads for tandem:
   P1 = 12.5 kips
   P2 = 12.5 kips
   P3 = N/A



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Segmental Box Girder Analysis to design

- Transverse analysis:
  - Find critical transverse moments (k-ft/ft) in slab, webs, and bottom floor.
  - If deck resultant stress is greater than allowable tensile stress, add transverse tendon(s), increase the thickness of component(s), etc.
  - Design web reinforcements from resultant moments of web
  - Design transverse reinforcement of deck from resultant moments of deck

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Segmental Box Girder Analysis to design

- Longitudinal analysis:
  - Use any time-dependent analysis software (for example BD2, RM, CSI, LARSA-4D, etc.) for the analysis
  - Normally a spine model is sufficient with pre-calculated segment cross-sectional properties (I, A, J, CG, etc.) and CG of tendons.
  - Things considered at each expected segment erection stage during design:
    - Time-dependent (creep & shrink) analysis is the function of segment casting dates, time that a segment is erected. Use AASHTO or CEB-FIP creep & shrinkage model.
    - Date that each cantilever tendon added
    - Date that each top-span tendon added
    - Date that each bottom-span tendon added
    - Date that each lock-in force developed (for example: removal of a lifter after closure pour)
    - Date of each continuity tendon added after a closure pour

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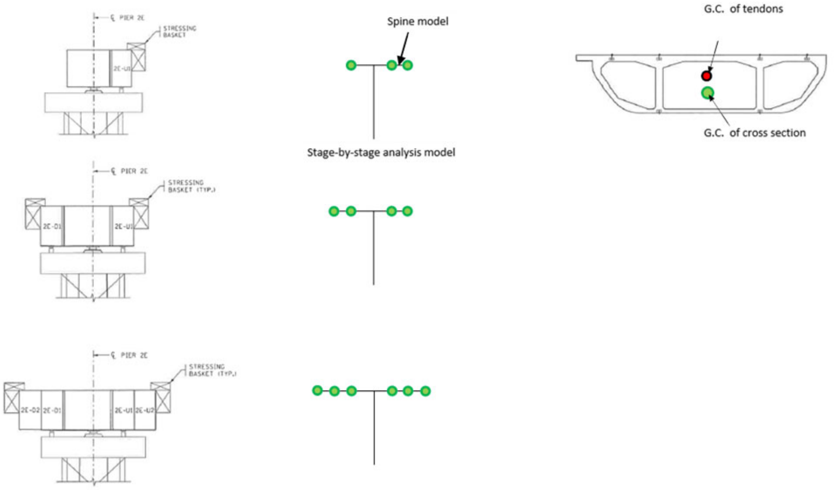
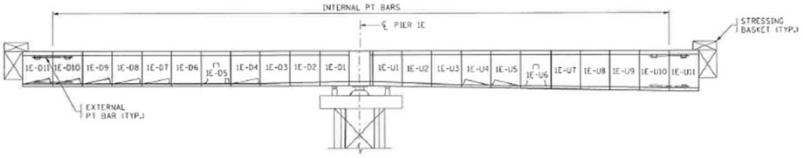
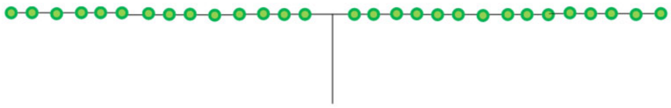
Typical Stage-by-stage Analysis

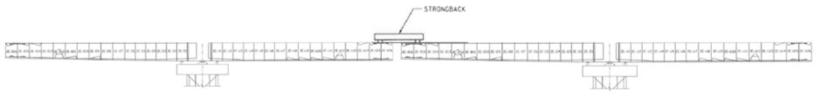


Typical Stage-by-stage Analysis





Typical Stage-by-stage Analysis



Closure pour

Consider lock-in force effect

Segmental Box Girder Analysis to design

- Longitudinal analysis:
  - Consider construction load, wind load at each stage construction
  - After stage-by-stage analysis is complete, perform analysis of the “whole” bridge considering temperature(both uniform temp. and temp. gradient), wind, LL (influence line), and tower/pier settlement.
  - Temperature analysis is to find the longitudinal stress due to temperature change

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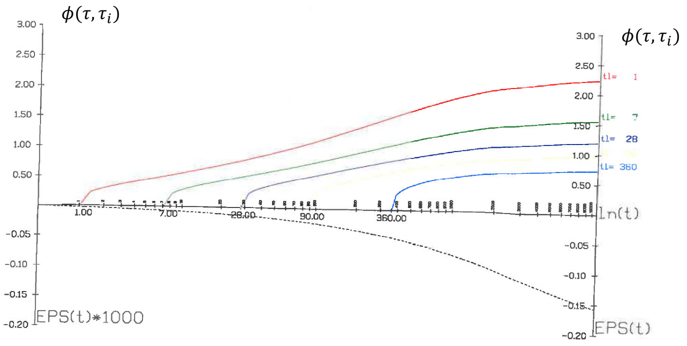
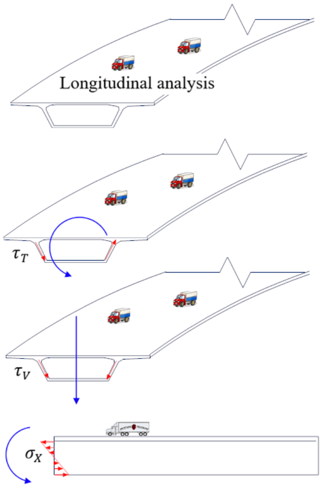
Segmental Box Girder Analysis to design

- Longitudinal analysis:
  - Quickly check the $\phi(\tau, \tau_i)$ at 10,000 days



Segmental Box Girder Analysis to design

- Web shear stress
  - Due to total torsion
  - Due to total shear force
- Web normal stress
  - Due to total bending
  - Due to P/T tendons



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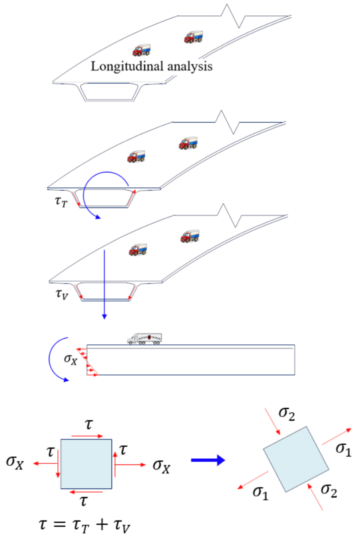
Web Principal Tensile Stress Check at Service Loads

- After longitudinal analysis is complete:
  - Combine shear stresses in the web:
    - $\tau = \tau_V + \tau_T$ (services I and III)
  - Find longitudinal axial stress in the web (σ_x)
  - From FEM software (or Mohr circle), find principal tensile stress σ_1
  - Check $\sigma_1$with allowable principal stress from codes (AASHTO limits it to $3.5\sqrt{f_c'}$ psi)



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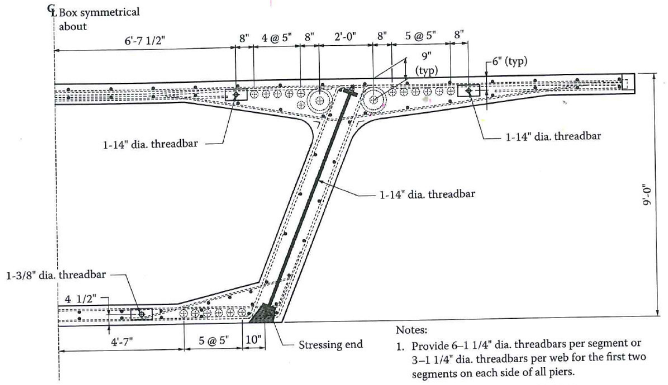
Web Principal Tensile Stress Check at Service Loads

- If it does not meet code requirement, increase thickness of web or add web PT bars
- Repeat analysis, until code requirement is met



Notes:

1. Provide 6–1 1/4" dia. threadbars per segment or 3–1 1/4" dia. threadbars per web for the first two segments on each side of all piers.
2. Stress each bar to 75% of G.U.T.S.

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Design Strength Limit State (I and IV) check

- Transverse flexural moments of deck
  - Make sure $\varphi M_n > \sum_i \gamma_i M_i$
- Longitudinal flexural moments of deck
  - Make sure $\varphi M_n > \sum_i \gamma_i M_i$
- Nominal Shear capacity check (V_n) at web(s)
  - Based on AASHTO 5.7.3.3
- Nominal Torsion capacity check (T_n)
  - Based on AASHTO 5.7.3.6.2; Make sure cross section $T_u \leq \phi T_n$
- Check combined Shear & Torsion stress:
  - Based on AASHTO 5.12.5.3:
  - $\left(\frac{V_u}{b_v d}\right) + \left(\frac{T_u}{2A_0 b_e}\right) \leq 0.474\sqrt{f_c'}$

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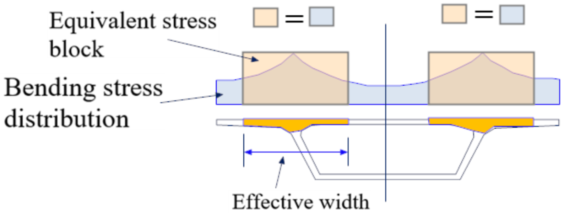
Slab equivalent width considering shear lag effect

- In calculating design capacity (M_n) of composite section,
  - Find bending stress distribution from FEA considering shear lag effect
  - Obtain equivalent width based on equivalent uniform stress block



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Learning Outcomes Review

By the end of this lesson, you should be able to:

- Explain the pitfalls of using FEA model results in design
- Process FEA results into design demand forces by integrating stresses
- Evaluate construction interface demand forces by FEA
- Perform cross-frame modeling & design
- Describe segmental bridge analysis and design
- Calculate slab equivalent width considering shear lag effect

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- Software demonstration of slicing feature to convert FEA stresses to design data.

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Next:
Course Evaluation/Wrap-up

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Course Closeout

Workshop Outcomes

- Describe the applications of refined analysis methods
- Select an appropriate refined analysis method for given bridge design and analysis scenarios
- Explain general steps and key parameters building a good FEA model for common bridges.
- Verify FEA results and describe the importance of validation efforts
- Describe the proper procedures for applying loads to a FEA bridge model.
- Explain how to translate FEA results into design input and code compliance.

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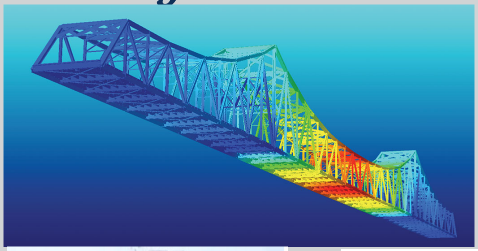
Course Evaluation

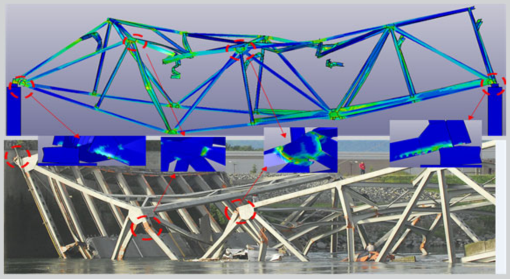
We value everyone’s input and feedback on this pilot training. Please fill out and submit the evaluation form thru the link provided in the chat box.

Final Thought

The boundary of structural engineering is unlimited, so is the world of FEA.

“Once you know more, you’ll find out you know less”





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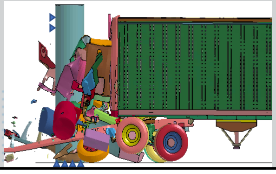

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Questions & Comments?

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Thank You!